THE UNIVERSITY OF ALBERTA

# COMPARISON THEOREMS FOR RICCATI DIFFERENTIAL EQUATIONS IN A $C^*$-ALGEBRA

BY

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A THESIS

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To my father and mother.

ABSTRACT

Let H be a real Hilbert space and $\mathcal{B}(H)$ be the $C^*$-algebra of bounded linear operators on H with the usual operator norm. This thesis is concerned with the self-adjoint $C^*$-valued Riccati differential equation

$$R[X](t) \equiv X'(t) + A^*(t)X(t) + X(t)A(t)$$
$$+ X(t)B(t)X(t) + C(t) = 0. \qquad (*)$$

We shall derive comparison theorems for $(*)$ under the assumption that A, B and C are continuous functions from some real interval J into $\mathcal{S}$, the subspace of $\mathcal{B}(H)$ consisting of self-adjoint operators. Our main motivation for this study is the tremendous amount of research that has been done on scalar and matrix Riccati differential equations which occur not only in analysis but in control theory, dynamic programming and other fields in engineering.

# PREFACE

Chapter I is devoted to background information and preliminary results. In section 1.1 we present the basic terminology and a list of symbols that will be used throughout the thesis. Section 1.2 deals with integration and differentiation of B-valued functions. In sections 1.3 and 1.4 we state a few basic facts concerning Banach algebras and self-adjoint operators in a Hilbert space, respectively. In section 1.5 we study the properties of positive operators. An example illustrates the fact that some well known results about positive operators in a finite dimensional Hilbert space no longer hold if $H$ is infinite dimensional. In section 1.6 we present theorems on the existence, uniqueness, and dependence on initial conditions of solutions of differential equations in Banach spaces. In section 1.7 we apply the results of section 1.6 to the Riccati differential equation $(*)$. In section 1.8 we study the initial value problem $X' = A(t)X$, $X(t_o) = X_o$, where $A$ is a continuous $C^*$-valued function. If $t_o \varepsilon (a,b)$, we show that this IVP has a unique solution $X(t)$ throughout $(a,b)$ and that $X(t)$ is non-singular on $(a,b)$ iff $X_o$ is non-singular. With the exception of some theorems and examples in sections 1.4, 1.5 and 1.8, the rest of Chapter I is completely expository and can be found in the literature.

The main results of this thesis are in Chapters II, III and IV. In Chapter II we study solutions of the Riccati differential equation $R[X](t) = 0$ and solutions of the Riccati differential inequalities obtained by replacing the equality by some

inequality. We generalize the standard comparison theorems that are known for matrix Riccati differential equations to the $C^*$-algebra case without any additional assumptions.

In Chapter III we give comparison theorems for Riccati differential equations of the form $X' + X^2 + Q(t) = 0$ where the comparisons are of integral type. The proofs for the matrix case make use of the fact that if $X(t)$ is a continuous matrix function such that $X(t_o) > 0$, then $X(t) > 0$ for all $t$ in some neighbourhood of $t_o$. This is not true if $H$ is infinite dimensional as demonstrated by an example in section 1.5. It is found, however, that these comparison theorems can be generalized to the $C^*$-algebra case by imposing a condition on the spectrum of the initial value. Other comparison theorems are also presented where, through the use of limit arguments, this additional assumption is not needed.

In Chapter IV we give examples and combine the results of Chapters II and III to obtain additional comparison theorems. Some of the theorems of this chapter give conditions under which $(*)$ has solutions that are positive throughout an unbounded interval, and hence they can be thought of as being non-oscillation theorems.

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# CHAPTER I

## INTRODUCTION AND PRELIMINARY RESULTS

### §1.1 Definitions and List of Symbols

In this section we shall define the spaces that will be of interest and present the basic terminology used throughout the thesis.

Let $F$ stand for either $\mathcal{R}$ or $\mathcal{C}$, the field of real and complex numbers, respectively. A complete normed space over the scalar field $F$ is called a Banach space (B-space) over $F$. A complete inner product space over $F$ is called a Hilbert space over $F$. The symbols $\| \ \|$ and $(,)$ shall denote the norm and inner product, respectively.

Definition. An algebra over $F$ is a vector space $X$ over $F$ in which to each ordered pair $(x,y) \in X \times X$ there corresponds an element $xy$ ("$x$ times $y$") of $X$ subject to the following axioms:

(i) $(xy)z = x(yz)$

(ii) $(x+y)z = xz + yz, \quad x(y+z) = xy + xz$

(iii) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$

for all $x,y,z$ in $X$ and $\alpha,\beta \in F$.

Definition. $X$ is a Banach algebra over $F$ if it is a B-space and an algebra over $F$ that contains a unit element $e$ such that $\|e\| = 1$, $xe = ex = x$ for all $x \in X$, and the inequality $\|xy\| \leq \|x\| \|y\|$ is satisfied for all $x, y \in X$.

Definition. If $X$ is an algebra over $F$ then an involution on $X$ is

a mapping $x \to x^*$ of $X$ into $X$ which satisfies

(i) $x^{**} = x$

(ii) $(xy)^* = y^*x^*$

(iii) $(\alpha x+\beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^*$

for all $x, y \in X$ and $\alpha, \beta \in F$. Here $\bar{\alpha}$ denotes the complex conjucate of $\alpha \in F$.

Definition. $X$ is said to be a C*-algebra over $F$ if it is a Banach algebra over $F$ in which $\|xx^*\| = \|x\|^2$ for all $x \in X$.

It follows immediately from this definition that an element $x$ of a C*-algebra satisfies the inequality $\|x\| \leq \|x^*\| \leq \|x^{**}\| = \|x\|$ and hence we must have $\|x\| = \|x^*\|$.

If the word 'real' or 'complex' prefaces a space it indicates the scalar field for that space.

We now present a few relevant and well known results from functional analysis (see e.g. Rudin [24]). If $X$ is a B-space then the space $\mathcal{B}(X)$ of bounded linear operators on $X$ with the usual operator norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

is a Banach algebra. In addition, if $H$ is a Hilbert space and $A \in \mathcal{B}(H)$ then there exists a unique $A^* \in \mathcal{B}(H)$ (called the adjoint of A) such that $(Ax,y) = (x,A^*y)$ for all $x, y \in H$. The mapping $A \to A^*$ is an involution on $\mathcal{B}(H)$ and $\mathcal{B}(H)$ is a C*-algebra relative to this involution.

An element $A$ of $\mathcal{B}(H)$ is said to be <u>self-adjoint</u> if $A = A^*$. It is clear that any closed self-adjoint subalgebra of $\mathcal{B}(H)$ is also a C*-algebra. Every C*-algebra can, in fact, be shown to be isometrically isomorphic to such an algebra (J. Dixmier [6], p. 45). It is in $\mathcal{B}(H)$, the most representative example of a C*-algebra, that we shall study Riccati differential equations.

## List of Symbols

| | |
|---|---|
| $A^*$ | Adjoint of $A$ |
| $A^{-1}$ | Inverse of $A$ |
| $I$ | Identity operator |
| $J$ | An interval in $\mathcal{R}$ |
| $\rho(A)$ | The resolvent set of $A$ |
| $\sigma(A)$ | The spectrum of $A$ |
| $\Vert\ \Vert$ | Norm |
| $(\ ,\ )$ | Inner product |
| $\square$ | Denotes end of proof |

<u>Symbols for particular spaces</u>. These symbols will be used throughout the thesis to mean the following:

| | |
|---|---|
| $\phi$ | The empty set |
| $\mathcal{R}$ | The real field |
| $\mathcal{C}$ | The complex field |
| $X$ | A <u>real</u> B-space |
| $H$ | A <u>real</u> Hilbert space |
| $\mathcal{B}(X)$ | The real Banach algebra of bounded linear operators on $X$. |

$\mathcal{B}(H)$ The real C*-algebra of bounded linear operators on H.

$\mathcal{S}$ The closed subspace of $\mathcal{B}(H)$ consisting of self-adjoint (symmetric, hermitian) operators

C[U,V] The space of continuous functions from U into V

The letter X will sometimes represent an operator. It will be obvious from the context whether X is a bounded operator or a real B-space.

We have taken the scalar field to be $\mathcal{R}$ throughout the thesis since $\mathcal{S}$ then is a Banach space, a fact that we shall need in the future. Nevertheless, all results of Chapter I (with the exception of Theorem 1.4.1) are valid even when the scalar field is complex. Also, the results of section 1.2 hold even if X is just a normed space.

## §1.2 Integration and Differentiation in B-spaces

Let J be a real interval and X a real B-space. Elements of the space C[J,X] are called abstract functions. The calculus of abstract functions is the topic of this section. We shall present the results that are of interest to us without any proofs. The proofs are very straightforward generalizations of those for the scalar case. This topic has been extensively dealt with by a variety of authors. In this regard we refer to the texts of J. Dieudonne [4], L.V. Kantorovich and G.P. Akilov [15], G.E. Ladas and V. Lakshmikantham [17], J.T. Schwartz [25], and G.E. Shilov [26].

### Continuity and Differentiability

The usual definitions of boundedness, continuity, differentiability, and so on apply for abstract functions. For example, the

abstract function $x:J \to X$ is continuous at the point $t_o \in J$ if $\| x(t)-x(t_o) \| \to 0$ as $t \to t_o$ and continuous on $J$ if it is continuous at each point of $J$. The convergence here, as throughout the rest of the thesis, is relative to the norm topology.

Abstract functions behave like ordinary functions in many respects. For example, continuous abstract functions map compact sets into compact sets and hence are bounded on compact sets. Also, a continuous abstract function is uniformly continuous on a compact set. For convenience the word abstract will be left out from now on.

$x(t)$ is said to be differentiable at a point $t_o \in J$ if the limit

$$x'(t_o) = \lim_{h \to o} \frac{x(t_o+h)-x(t_o)}{h} ,$$

called the derivative of $x(t)$ at $t = t_o$, exists in $X$. If $t_o$ is an endpoint of $J$ then the appropriate one-sided limit is taken. $x(t)$ is differentiable on $J$ if it is differentiable at every point of $J$. The following properties of differentiation are the exact analogues of those for scalar functions:

1) If $x(t)$ is differentiable at $t_o$ then it is continuous at $t_o$.

2) If $x \in C[[a,b],X]$ and $\| x'(t) \| \leq K$ on $[a,b]$ then

$$\| x(b)-x(a) \| \leq K(b-a).$$

3) If $x(t)$ is differentiable on $[a,b]$ then

$$x(b) - x(a) \in (b-a)\, \overline{co}\, \{x'(t):t\in[a,b]\}$$

where $\overline{co}$ denotes the closed convex hull.

4) If $x(t)$ and $y(t)$ are differentiable on $J$ then so is $\alpha x(t) + y(t)$, $\alpha \in \mathcal{R}$, and $[\alpha x(t)+y(t)]' = \alpha x'(t) + y'(t)$.

5) If $X$ is a Banach algebra, $x(t)$ and $y(t)$ are differentiable on $J$, then so is $x(t)y(t)$ and

$$[x(t)y(t)]' = x(t)y'(t) + x'(t)y(t).$$

6) If $x(s)$ is differentiable on $J$ and $s(t)$ is a real function with values in $J$ and differentiable at $t_o$ then the composite function $y(t) = x(s(t))$ is differentiable at $t_o$ and

$$y'(t_o) = x'(s(t_o))s'(t_o).$$

7) If $x(t)$ is differentiable, $X'$ is also a B-space, and $A$ is a bounded linear operator from $X$ into $X'$ then

$$[Ax(t)]' = Ax'(t).$$

Also, if $A = A(t)$ is differentiable then

$$[A(t)x]' = A'(t)x, \quad x \in X.$$

8) If $H$ is a Hilbert space and $A \in C[J,\mathcal{B}(H)]$ is differentiable on $J$ then

$$[(A(t)x,y)]' = (A'(t)x,y)$$

for all $x, y \in H$.

<u>The Riemann Integral</u>

Let $x:[a,b] \to X$ be an abstract function. For any partition

$$\pi = \{a = t_o \leq \tau_o \leq t_1 \leq \ldots \leq t_{n-1} \leq \tau_{n-1} \leq t_n = b\}$$

with diameter

$$|\pi| = \max_i \Delta t_i \qquad (\Delta t_i = t_{i+1} - t_i)$$

we can form the Riemann sum

$$S_\pi = \sum_{i=o}^{n-1} x(\tau_i)\Delta t_i$$

If there exists an element $I$ of $X$ such that $S_{\pi_n} \to I$ for every sequence of partitions $\{\pi_n\}$ for which $|\pi_n| \to 0$ we say that $I$ is the integral of $x(t)$ over $[a,b]$ and write

$$I = \int_a^b x(t)dt.$$

As in the scalar case we have the following properties of the integral:

1) If $x \in C[[a,b],X]$ then $\int_a^b x(t)dt$ exists.

2) $\int_a^b x(t)dt = - \int_b^a x(t)dt$ if at least one of the integrals exist.

3) $\int_a^b x(t)dt = \int_a^c x(t)dt + \int_c^b x(t)dt$ provided that the integral on the left exists.

4) $\int_a^b [\alpha x(t)+y(t)]dt = \alpha \int_a^b x(t)dt + \int_a^b y(t)dt$, $\alpha$ is real, if the integrals on the right exist.

5) $\|\int_a^b x(t)dt\| \leq \int_a^b \|x(t)\| dt$ if $x \in C[[a,b],X]$.

6) $\int_a^b x_o dt = (b-a)x_o$

7) If $x(t)$ is integrable on $[a,b]$ then

$$\frac{1}{b-a}\int_a^b x(t)dt \; \varepsilon \; \overline{co}\,\{x(t) : t \varepsilon [a,b]\}$$

8) If $X$ is a Banach algebra and $x \; \varepsilon \; C[[a,b],X]$ then

$$\int_a^b x(t)ydt = \left(\int_a^b x(t)dt\right)y$$

for $y \; \varepsilon \; X$ and similarly for left multiplication.

9) If $x \; \varepsilon \; C[[a,b],X]$, $X'$ is also a B-space, and $A$ is a bounded linear operator from $X$ into $X'$ then

$$\int_a^b Ax(t)dt = A\left(\int_a^b x(t)dt\right).$$

Also, if $A = A(t)$ is continuous on $[a,b]$ then

$$\int_a^b A(t)xdt = \left(\int_a^b A(t)dt\right)x$$

for $x \; \varepsilon \; X$.

10) If $x \; \varepsilon \; C[[a,b],X]$ then

$$\left[\int_a^t x(s)ds\right]' = x(t)$$

for all $t \; \varepsilon \; [a,b]$.

11) If $x(t)$ is continuously differentiable on $[a,b]$ then

$$x(t) = x(a) + \int_a^t x'(s)ds$$

for all $t \in [a,b]$.

12) If $X$ is a Banach algebra and $x(t)$ and $y(t)$ are continuously differentiable on $[a,b]$ then

$$\int_a^b x(t)y'(t)\,dt = x(t)y(t)\Big|_a^b - \int_a^b x'(t)y(t)\,dt$$

13) If $u(t)$ is a real, continuously differentiable function on $[a,b]$ and $x$ is a continuous abstract function on $\{u(t): t\in[a,b]\}$ then

$$\int_{u(a)}^{u(b)} x(s)\,ds = \int_a^b x(u(t))u'(t)\,dt.$$

Improper Integrals

Let $x: [a,b) \to X$ be an abstract function not defined at $b \leq \infty$. If $x(t)$ is integrable on every interval $[a,c] \subset [a,b)$ then the improper integral of $x$ over $[a,b)$ is defined as

$$\lim_{\varepsilon\to 0} \int_a^{b-\varepsilon} x(t)\,dt \quad \text{if } b < \infty$$

and

$$\lim_{N\to\infty} \int_a^N x(t)\,dt \qquad \text{if } b = \infty$$

provided that the limit exists.

Sequences and Series of Abstract Functions

Let $x_n(t)$ be a sequence of abstract functions. We say $x_n(t)$

converges to $x(t)$ on $[a,b]$ if $\lim_n \| x_n(t)-x(t) \| = 0$ for each $t \varepsilon [a,b]$. Moreover, we say $x_n(t)$ converges uniformly to $x(t)$ on $[a,b]$ if

$$\lim_{n\to\infty} \sup_{t\varepsilon[a,b]} \| x_n(t)-x(t) \| = 0.$$

The following results from elementary analysis hold for sequences of abstract functions:

1) The limit of a uniformly convergent sequence of continuous functions is itself continuous.

2) If $x_n(t)$ is a sequence of functions integrable on $[a,b]$ which converge uniformly on $[a,b]$ to $x(t)$ then $x(t)$ is also integrable and

$$\lim_{n\to\infty} \int_a^t x_n(s)ds = \int_a^t x(s)ds$$

holds uniformly for all $t \varepsilon [a,b]$.

3) Let $x_n(t)$ be a sequence of continuously differentiable functions on $[a,b]$ that converge for at least one point in $[a,b]$. Suppose that the sequence $x_n'(t)$ converges uniformly on $[a,b]$ to $y(t)$. Then the sequence $x_n(t)$ converges uniformly on $[a,b]$ to a continuously differentiable function $x(t)$ with derivative

$$x'(t) = \lim_{n\to\infty} x_n'(t) = y(t)$$

We say that the infinite series $\sum_{n=1}^{\infty} x_n(t)$ converges on $[a,b]$ if the sequence of partial sums $S_n(t) = \sum_{j=1}^{n} x_j(t)$ converges on

[a,b]. The preceding two results give sufficient conditions for term-by-term integration and differentiation of a series of abstract functions.

The results of this section will be used implicitly in the succeeding chapters.

## §1.3 Non-singular Elements of a Banach Algebra

We consider the Banach algebra $\mathcal{B}(X)$ where $X$ is a real B-space. We say that $A \in \mathcal{B}(X)$ is non-singular if there exists an element $A^{-1}$ in $\mathcal{B}(X)$, called the inverse of $A$, such that $AA^{-1} = A^{-1}A = I$ where $I$ is the unit in $\mathcal{B}(X)$.

We show now that the unit ball about $I$ consists of non-singular elements.

THEOREM 1.3.1 *If* $A \in \mathcal{B}(X)$ *and* $\|A\| < 1$ *then* $I - A$ *is non-singular.*

Proof. Let $S_n = \sum_{j=o}^{n} A^j$ (with $A^o = I$).
From

$$\|S_n - S_m\| = \|\sum_{n+1}^{m} A^j\| \leq \sum_{n+1}^{m} \|A\|^j$$

follows the fact that $S_n$ converges to an element $S$ of $\mathcal{B}(X)$. Note that $(I-A)S_n = S_n(I-A) = I - A^{n+1}$ and hence $(I-A)S = S(I-A) = I$. That is, $I - A$ is non-singular with inverse $S$. □

COROLLARY 1.3.2 *If* $A \in \mathcal{B}(X)$ *and* $\|A-I\| < 1$ *then* $A$ *is non-singular.*

Proof. This follows immediately from the above theorem since

$A = I - (I-A)$ and $\|I-A\| < 1$. □

The following theorem shows that the set of non-singular elements is open in $\mathcal{B}(X)$.

<u>THEOREM 1.3.3</u> *If* $A_o \in \mathcal{B}(X)$ *is non-singular and* $\|A-A_o\| < \|A_o^{-1}\|^{-1}$ *then* $A$ *is non-singular also.*

<u>Proof.</u> We have

$$A = A_o - (A_o-A) = A_o[1-A_o^{-1}(A_o-A)].$$

$[1-A_o^{-1}(A_o-A)]$ is non-singular by Theorem 1.3.1 since $\|A_o^{-1}(A_o-A)\| < 1$. Thus $A$ is non-singular being the product of two non-singular elements. □

It is also easily shown (Rudin [24]) that the set of non-singular elements of $\mathcal{B}(X)$ is a group and that the mapping $A \to A^{-1}$ is a continuous map of this group onto itself.

## §1.4 <u>Spectral Theory of Self-Adjoint Operators</u>

Let $H$ be a real Hilbert space and $\mathcal{S}$ the subset of $\mathcal{B}(H)$ consisting of self-adjoint operators. We have the following result concerning $\mathcal{S}$.

<u>THEOREM 1.4.1</u> $\mathcal{S}$ *is a Banach space (with the same norm as* $\mathcal{B}(H)$*).*

<u>Proof.</u> Since the scalar field is real it is easily verified that $\mathcal{S}$ is a subspace of $\mathcal{B}(H)$. Now suppose that $\{A_n\}$ is a sequence of self-adjoint operators that converge in $\mathcal{B}(H)$ to $A$. That $A$ is also self-

adjoint follows from the fact that

$$(Ax,y) = \lim_n (A_n x,y) = \lim_n (x,A_n y) = (x,Ay)$$

for all $x, y \in H$. Thus $\mathcal{S}$ is a closed subspace of $\mathcal{B}(H)$ and is therefore a B-space. $\square$

$\mathcal{S}$ is not a Banach algebra since the product of two arbitrary self-adjoint operators is not necessarily self-adjoint. We have the following result however:

<u>THEOREM 1.4.2</u> *If* $A$ *and* $B$ *are self-adjoint operators that commute (i.e.* $AB = BA$*) then* $AB$ *is also self-adjoint.*

<u>Proof</u>. Follows immediately from

$$(AB)^* = B^*A^* = BA = AB. \qquad \square$$

If $A \in \mathcal{S}$ then it can be shown (Plesner [21], p. 199) that the norm of $A$ can be written as

$$\|A\| = \sup_{\|x\|=1} |(Ax,x)|.$$

The greatest lower bound and the least upper bound of a self-adjoint operator $A$ are defined by

$$m(A) = \inf_{\|x\|=1} (Ax,x) \quad \text{and} \quad M(A) = \sup_{\|x\|=1} (Ax,x),$$

respectively. It then follows that

$$\|A\| = \max\{|m(A)|, |M(A)|\}.$$

If $A \in \mathcal{B}(H)$ then we say that $\lambda$ is in the resolvent set $\rho(A)$ of $A$ if $\lambda I - A$ is a bijection. Note that by the open mapping theorem $\lambda I - A$ is a bijection iff it has a bounded inverse. If $\lambda \notin \rho(A)$ then $\lambda$ is said to be in the spectrum $\sigma(A)$ of $A$.

A direct consequence of Theorem 1.3.3 is that $\rho(A)$ is open and hence $\sigma(A)$ is closed. From Theorem 1.3.1 it also follows that $\sigma(A) \subset [-\|A\|, \|A\|]$. Thus $\sigma(A)$ is a compact set. Since our scalar field is real it is not necessarily true that every bounded operator has a nonempty spectrum. However for self-adjoint operators we have the following result:

THEOREM 1.4.3 *If* $A \in \mathcal{S}$ *then* $\sigma(A) \subset [m(A), M(A)]$. *Furthermore,* $m(A)$ *and* $M(A)$ *belong to* $\sigma(A)$.

Proof. See Liusternik and Sobolev [18], p. 148. □

Thus every self-adjoint operator has a non-empty spectrum. We now show that $m(A)$ and $M(A)$ are continuous functions of $A$.

THEOREM 1.4.4 $m$ *and* $M$ *belong to* $C[\mathcal{S}, \mathbb{R}]$.

Proof. Let $A$ and $A_o$ be self-adjoint operators and $x$ an arbitrary element of $H$ with $\|x\| = 1$. Then

$$\begin{aligned}(Ax,x) &= (A_o x,x) + \left((A-A_o)x,x\right) \\ &\leq (A_o x,x) + \|A-A_o\| \end{aligned} \tag{1}$$

and hence $m(A) \leq m(A_o) + \|A-A_o\|$. Also,

$$\begin{aligned}(A_o x,x) &= (Ax,x) + ((A_o - A)x,x) \\ &\leq (Ax,x) + \|A_o - A\| \qquad (2)\end{aligned}$$

and hence $m(A_o) \leq m(A) + \|A_o - A\|$. Therefore $|m(A) - m(A_o)| \leq \|A - A_o\|$ and we conclude $m$ is continuous on $\mathcal{S}$.

Similarly, from equations (1) and (2) we obtain

$$M(A) \leq M(A_o) + \|A - A_o\|$$

and

$$M(A_o) \leq M(A) + \|A_o - A\|,$$

respectively. Thus $|M(A) - M(A_o)| \leq \|A - A_o\|$ and $M$ is also continuous on $\mathcal{S}$. □

§1.5 Positive Operators

We now define a partial ordering on $\mathcal{B}(H)$.

Definition. A positive operator, $A > 0$, is a self-adjoint operator that satisfies $(Ax,x) > 0$ for all $x \in H$, $x \neq 0$. If the strict inequality is replaced by an inequality we say that $A$ is non-negative, $A \geq 0$.

Non-negative operators have the following properties:

1) If $A \geq 0$, $B \geq 0$, and $\alpha$ is a non-negative real number then $\alpha A + B \geq 0$.

2) If $A \geq 0$ and $-A \geq 0$ then $A = 0$.

That is, the set of non-negative operators is a strict positive

cone in $\mathcal{B}(H)$.

Definition. For $A, B \in \mathcal{B}(H)$ we say $A \geq B$ $(A > B)$ if $A - B \geq 0$ $(A - B > 0)$. A self-adjoint operator $C$ is said to be non-positive (negative) if $0 \geq C$ $(0 > C)$.

If $A \geq B$ $(A > B)$ then we also say $B \leq A$ $(B < A)$. From the discussion above it is seen that "$\geq$" and "$\leq$" are partial orders on $\mathcal{B}(H)$. Note that by our definition, a non-negative operator is required to be self-adjoint. This condition is not always imposed but we do so since otherwise "$\geq$" is no longer a partial order on $\mathcal{B}(H)$. For example, if $H = \mathcal{R}^2$ and $A$ is the operator representing rotation by $\frac{\pi}{2}$ then $(Ax,x) = 0$ for all $x \in H$. Thus $A \geq 0$ and $A \leq 0$ but $A \neq 0$ and hence "$\geq$" is not a partial order. Imposing the condition of self-adjointness removes this problem.

We now show that the set of non-negative operators is a closed strict cone in $\mathcal{B}(H)$.

LEMMA 1.5.1 *If* $x \in H$ *then the function* $f_x(A) = (Ax,x)$ *belongs to* $C[\mathcal{B}(H),\mathcal{R}]$.

Proof. If $A, A_o \in \mathcal{B}(H)$ then

$$\begin{aligned} |f_x(A)-f_x(A_o)| &= |((A-A_o)x,x)| \\ &\leq \|A-A_o\| \, \|x\|^2 \end{aligned}$$

and hence $f_x \in C[\mathcal{B}(H),\mathcal{R}]$. $\square$

THEOREM 1.5.2 *The set of non-negative operators is closed in* $\mathcal{B}(H)$.

Proof. Suppose $A_o \not\geq 0$. Then there exists a $x_o \in H$, $x_o \neq 0$ such that $f_{x_o}(A_o) < 0$. Thus by our lemma, $f_{x_o}(A) < 0$ for all $A$ in some neighbourhood of $A_o$. This shows that the set $\{A : A \not\geq 0\}$ is open in $\mathcal{B}(H)$ and hence the set $\{A : A \geq 0\}$ is closed in $\mathcal{B}(H)$. □

COROLLARY 1.5.3 *If* $A : \mathcal{R} \to \mathcal{B}(H)$ *is continuous at* $t_o$ *and* $A(t_o) \not\geq 0$ *then* $A(t) \not\geq 0$ *for all* $t$ *in some neighbourhood of* $t_o$.

COROLLARY 1.5.4 *If* $A \in C[[a,c],\mathcal{B}(H)]$ *and* $A(t) > 0$ *on* $[a,c)$ *then* $A(c) \geq 0$.

The corollaries follow immediately from Theorem 1.5.2. Riccati differential equations involving abstract functions like those in the above corollaries is the topic of this thesis. In particular, we shall be dealing almost exclusively with abstract functions from $\mathcal{R}$ into $\mathcal{S}$. The following result gives a condition under which such an abstract function is bounded.

THEOREM 1.5.5 *If* $A, B \in C[[a,c],\mathcal{S}]$, $X \in C[[a,c),\mathcal{S}]$ *and*

$$A(t) \geq X(t) \geq B(t)$$

*for all* $t \in [a,c)$ *then* $X(t)$ *is bounded on* $[a,c)$.

Proof. For any $x \in H$, $\|x\| = 1$ and $t \in [a,c)$ we have

$$(X(t)x,x) \leq (A(t)x,x) \leq \sup_{[a,c]} \|A(t)\|$$

and hence $M(A(t)) \leq \sup_{[a,c]} \|A(t)\|$ for all $t \in [a,c)$. Also,

$$(X(t)x,x) \geq (B(t)x,x) \geq -\sup_{[a,c]} \|B(t)\|$$

and hence $m(A(t)) \geq -\sup_{[a,c]} \|B(t)\|$ for all $t \in [a,c]$. Thus

$$\|X(t)\| = \max\{|m(A(t))|, |M(A(t))|\}$$

$$\leq \max\left\{\sup_{[a,c]} \|A(t)\|, \sup_{[a,c]} \|B(t)\|\right\}$$

for all $t \in [a,c)$. i.e. $X(t)$ is bounded on $[a,c)$. □

It is clear from the definition that $A \geq 0$ iff $m(A) \geq 0$. Also, if $m(A) > 0$ then $A > 0$ but not conversely as we will soon see. Thus $A \geq 0$, $A \not> 0$ implies that $m(A) = 0$. In fact, the next theorem characterizes non-negative operators that are not positive as those having zero as an eigenvalue.

<u>THEOREM 1.5.6</u> *If* $A \geq 0$ *and* $A \not> 0$ *then there exists a* $x_o \in H$, $x_o \neq 0$ *such that* $Ax_o = 0$.

<u>Proof</u>. See A. Plesner [21], p. 201. □

From Corollary 1.5.4 and Theorem 1.5.6 we obtain the following result.

<u>COROLLARY 1.5.7</u> *If* $A \in C[[a,c],\mathcal{B}(H)]$ $A(t) > 0$ *on* $[a,c)$ *and* $A(c) \not> 0$ *then there exists a* $x_o \in H$, $x_o \neq 0$, *such that* $A(c)x_o = 0$.

If $H$ is finite dimensional and $A_o > 0$ then $m(A_o) > 0$ and hence $A > 0$ for all $A$ in some neighbourhood of $A_o$ in $\mathcal{S}$. Thus the

set of positive operators is open in $S$. This is not true in an infinite dimensional space as the following example shows.

Example. Let $H = \ell^2$,

$$a_{ij}(t) = \delta_{ij}[(-1)^{j+1}t + \frac{1}{j}],$$

where $\delta_{ij}$ is the Kronecker delta, and $A(t) = (a_{ij}(t))$. That is, $A(t)$ is the infinite diagonal matrix $\operatorname{diag}\{(-1)^{n+1}t + \frac{1}{n}\}$. Since

$$\sum_{j=1}^{\infty} |a_{ij}(t)| \leq |t| + 1, \quad i = 1,2,3,\ldots$$

and

$$\sum_{i=1}^{\infty} |a_{ij}(t)| \leq |t| + 1, \quad j = 1,2,3,\ldots$$

then, by a theorem of Schur (A. Taylor [27], p. 328), $A(t)$ represents a bounded linear operator on $\ell^2$ for all $t$. Or more directly, if $x = (x_n) \in \ell^2$ and $\|\ \|_2$ denotes the $\ell^2$ norm then

$$\begin{aligned}\|A(t)x\|_2^2 &= \sum_{n=1}^{\infty}\left[(-1)^{n+1}tx_n + \frac{x_n}{n}\right]^2 \\ &= t^2\left(\sum_n x_n^2\right) + \left(\sum_n (\frac{x_n}{n})^2\right) \\ &\quad + 2t\left(\sum_n (-1)^{n+1}(\frac{x_n}{\sqrt{n}})^2\right) < \infty \quad \text{for all} \quad t \in \mathbb{R}\end{aligned}$$

and therefore $A(t)x \in \ell^2$. Also, $A(t)$ is self-adjoint for all $t$

since it is a diagonal matrix. We now show $A(t)$ is a bounded operator for all $t$. The spectrum of $A(t)$ is the closure of its diagonal elements and hence we obtain

$$\sigma[A(t)] = \{-t,+t\} \cup \{(-1)^{n+1}t+\frac{1}{n} : n=1,2,\ldots\} \tag{1}$$

By inspection of (1) we find

$$m(A(t)) = -|t| \tag{2}$$

and

$$\begin{aligned} M(A(t)) &= \max\{1+t,\tfrac{1}{2}-t\} \\ &= \begin{cases} 1+t, & t \geq -\tfrac{1}{4} \\ \tfrac{1}{2}-t, & t \leq -\tfrac{1}{4} \end{cases} \end{aligned} \tag{3}$$

From (2) and (3) we obtain

$$\begin{aligned} \|A(t)\| &= \max\{|m(A(t))|,|M(A(t))|\} \\ &= M(A(t)) \\ &= \begin{cases} 1+t, & t \geq -\tfrac{1}{4} \\ \tfrac{1}{2}-t, & t \leq -\tfrac{1}{4} \end{cases} \end{aligned}$$

This shows $A(t) \in S$ for all $t$. Now, for any real numbers $t$ and $t_1$ we have

$$\begin{aligned} &\|[A(t)-A(t_1)]x\|_2 \\ &= \|(\mathrm{diag}[(-1)^{n+1}(t_1-t)])x\|_2 \\ &= |t-t_1| \ \|x\|_2. \end{aligned}$$

Therefore $\| A(t)-A(t_1) \| = |t-t_1|$ and $A \in C[\mathcal{R},\mathcal{S}]$. At $t = 0$ we see that

$$(A(0)x,x) = \sum_n \frac{x_n^2}{n} > 0$$

for all $x \neq 0$ and hence $A(0) > 0$. It is clear from (2) however that $A(t) \not> 0$ for all $t \neq 0$.

In summary, $A \in C[\mathcal{R},\mathcal{S}]$ and $A(t) > 0$ iff $t = 0$. Therefore the set of positive operators is not an open subset of $\mathcal{S}$. However there is a subset of the set of positive operators that is open in $\mathcal{S}$ as the following theorem shows.

<u>THEOREM 1.5.8</u> *In any Hilbert space* $H$, *the set* $\{A > 0 : 0 \notin \sigma(A)\}$ *is an open subset of* $\mathcal{S}$.

<u>Proof</u>. If $A_o > 0$ and $0 \notin \sigma(A_o)$ then $m(A_o) > 0$ and hence $m(A) > 0$ for all $A$ in some neighbourhood of $A_o$ in $\mathcal{S}$ (by Theorem 1.4.4). Clearly, $A > 0$ and $0 \notin \sigma(A)$ for all operators $A$ in this neighbourhood. □

<u>COROLLARY 1.5.9</u> *If* $A:\mathcal{R} \to \mathcal{S}$ *is continuous at* $t_o$, $A(t_o) > 0$ *and* $0 \notin \sigma(A(t_o))$ *then* $A(t) > 0$ *for all* $t$ *in some neighbourhood of* $t_o$.

Finally, we present two last results about positive operators.

<u>THEOREM 1.5.10</u> *If* $A$ *and* $B$ *are positive (non-negative) operators that commute then* $AB$ *is also a positive (non-negative) operator.*

<u>Proof</u>. See A. Plesner [21], p. 201. □

An application of Theorem 1.5.10 is the following:

COROLLARY 1.5.11 *If* $A$ *and* $B$ *commute and* $B \geq A \geq 0$ $[B > A > 0]$ *then* $B^2 \geq A^2 \geq 0$ $[B^2 > A^2 > 0]$.

Proof. If $A$ and $B$ commute then so do $B + A$ and $B - A$ with $(B+A)(B-A) = (B-A)(B+A) = B^2 - A^2$. The corollary now follows immediately from Theorem 1.5.10. □

## §1.6 Ordinary Differential Equations in Banach Spaces

A great amount of research has been done recently and several texts published on the theory of ordinary differential equations in Banach spaces. In this regard we refer to the texts of K. Deimling [3], S.G. Krein [16], G.E. Ladas and V. Lakshmikantham [17], and R.H. Martin [19], and to the research papers of J. Dieudonne [5], G.J. Etgen and R.T. Lewis [8], T.L. Hayden and H.C. Howard [11], and E.S. Noussair [20].

Let $J$ be a real interval, $D$ a closed subset of a real Banach space $X$, and $f$ a function from $J \times D$ into $X$. We consider the initial value problem (IVP)

$$x' = f(t,x), \quad x(t_o) = x_o \tag{1}$$

where $(t_o,x_o) \in J \times D$. In this section we shall quote several theorems concerning the existence and uniqueness of solutions of (1) and their dependency on initial conditions.

In case $X = R^n$ it is well known (Peano's existence theorem) that continuity of $f$ in a neighbourhood of $(t_o,x_o)$ implies the existence of a local solution of (1). Ladas and Lakshmikantham [17,

p. 128] provide counterexamples that show this result cannot be generalized to infinite dimensional spaces. However a generalization of the classical Picard-Lindelof theorem is possible.

THEOREM 1.6.1 *Let* $B(c;r)$ *denote the closed ball of radius* $r$ *with center* $c$. *Let* $R = B(t_o;a) \times B(x_o;b)$ *and suppose that* $\|f(t,x)\| \leq M$ *on* $R$. *If* $f(t,x)$ *is continuous in* $t$ *for each fixed* $x$ *and* $\|f(t,x_1)-f(t,x_2)\| \leq K\|x_2-x_1\|$ *for* $(t,x_1)$, $(t,x_2) \in R$ *where* $M$, $K \geq 0$ *then* (1) *has a unique solution* $x(t)$ *on* $B(t_o;\alpha)$ *where* $\alpha = \min\{a, \frac{b}{M}\}$.

Proof. See Ladas and Lakshmikantham [17], p. 129. □

Now suppose that $J$ is a real interval and $D$ a closed subset of $X$ such that $(t_o,x_o) \in J \times D$. Suppose also that $x:J_x \to D$ and $y:J_y \to D$ are solutions of (1) on $J_x \subset J$ and $J_y \subset J$, respectively. We say that $y$ is a continuation of $x$ if $J_y \supset J_x$ and $y(t) = x(t)$ for all $t \in J_x$. We say the solution $x(t)$ of (1) is noncontinuable if it has no proper continuation. If "$x \leq y$" is defined to mean that $y$ is a continuation of $x$ then "$\leq$" is a partial order on solutions of (1). By use of Zorn's lemma, the following result can be proved.

THEOREM 1.6.2 *If* $f \in C[J\times D,X]$ *then every solution* $x(t)$ *of* (1) *has a noncontinuable continuation* $y(t)$.

Proof. See R.H. Martin [19], p. 199. □

Note that if (1) has a unique solution then it has exactly

one noncontinuable solution.

Let $x(t)$ be a solution of (1) and $y:J_y \to D$ be the noncontinuable continuation of $x(t)$. The interval $J_y \cap [t_o,\infty)$ is called the <u>right maximal interval of existence</u> of $x(t)$. We say that $y:J_y \cap [t_o,\infty) \to D$ is a solution of (1) that is <u>noncontinuable to the right</u>. A local solution to the right is a solution to (1) of the form $u:[t_o,t_o+\delta] \to D$, for some $\delta > 0$. Analagous definitions can be made for solutions to the left in an obvious manner.

For the case $X = R^n$ there is a "continuation of solutions" theorem that says if $f \in C[[t_o,t_o+a]\times X,X]$ then any solution $x(t)$ of (1) exists either on $[t_o,t_o+a]$ or on $[t_o,\delta)$ with $\delta \in (t_o,t_o+a)$ and $\|x(t)\| \to \infty$ as $t \to \delta^-$. Ladas and Lakshmikantham ([17], p. 131) give an illustrative counterexample to show that this result does not hold in infinite dimensional spaces. It does hold however if additional conditions are imposed.

<u>THEOREM 1.6.3</u> *Suppose that* $f \in C[J\times D,X]$, *and maps closed bounded sets into bounded sets. Suppose also that* (1) *has a local solution to the right for each* $(t_o,x_o) \in J \times D$ *with* $t_o$ *not a right endpoint of* $J$. *If* $x:J_x \to D$ *is a solution to the right of the* IVP (1) *that is noncontinuable to the right then either* $J_x = J \cap [t_o,\infty)$ *or* $J_x = [t_o,c)$, *for some* $c > t_o$, *and* $\|x(t)\| \to \infty$ *as* $t \to c^-$.

<u>Proof</u>. See Martin [19], p. 200. □

Theorem 1.6.3 with the obvious modifications also applies to solutions to the left.

An example of Dieudonne [5] shows that the condition $f$ map

bounded sets into bounded sets is a necessary condition for Theorem 1.6.3 to hold. We now examine the effect that perturbing the initial condition (and the differential equation itself) of (1) has on its solutions. The theorem that follows is a weaker form of one given by Martin in his text ([19], p. 222).

<u>THEOREM 1.6.4</u> *Let* $(t_o,x_o) \in J \times X$ *with* $t_o$ *not a right endpoint of* $J$ *and suppose the following conditions hold:*

(i) $\{f_n\}_o^\infty$ are continuous functions from $J \times X$ into $X$, where $f_o = f$.

(ii) The IVP (1) has the unique solution $u(t)$ whose domain contains $[t_o,b] \subset J$.

(iii) For each $R > 0$ there is a continuous function $\rho_R : J \to [0,\infty)$ such that

$$\| f_n(t,x) - f_n(t,y) \| \leq \rho_R(t) \| x-y \|$$

for all $(t,x)$, $(t,y) \in J \times X$ with $\| x \|$, $\| y \| \leq R$ and for all $n \geq 0$.

(iv) $f_n(t,x) \to f(t,x)$ uniformly for $(t,x)$ in each closed, bounded subset of $J \times X$.

Now suppose that $(t_n,x_n) \to (t_o,x_o)$ in $J \times X$ and let $u_n(t)$ be a noncontinuable solution of the IVP

$$u_n' = f_n(t,u_n), \quad u_n(t_n) = x_n \tag{2}$$

for $n = 1,2,3,\ldots$ . Then there exists a $n_o \geq 1$ such that the domain

of $u_n(t)$ contains $[t_n,b]$ for all $n \geq n_o$. Also, for any $\varepsilon > 0$ there exists a $n(\varepsilon) \geq n_o$ such that $\| u_n(t)-u(t) \| < \varepsilon$ for all $t \in [t_o,b] \cap [t_n,b]$ and all $n \geq n(\varepsilon)$.

Note that if $t_n = t_o$ for all $n$ then the sequence $\{u_n(t)\}_{n \geq n_o}$ converges to $u(t)$ uniformly on $[t_o,b]$.

Proof. See Martin [19], p. 222. □

A result identical to Theorem 1.6.4 holds if $u(t)$ is a solution of (1) on an interval of the form $[a,t_o]$. Combining the two results yield the following corollary:

COROLLARY 1.6.5 *Under the same conditions as in Theorem 1.6.4, if* $u(t)$ *is a unique solution of* (1) *whose domain contains* $[a,b]$ *with* $t_o \in [a,b] \subset J$, *then there is a subsequence* $\{u_{n_i}(t)\}$ *such that* $u_{n_i}(t) \to u(t)$ *uniformly on* $[a,b]$.

## §1.7 Riccati Differential Equations - Uniqueness of Solutions and Dependence on Initial Conditions

Let $J$ be a real interval, $H$ a real Hilbert space and $\mathcal{S}$ the subset of $\mathcal{B}(H)$ consisting of self-adjoint operators. We will apply the results of the previous section to the general Riccati differential equation

$$X' + A^*(t)X + XA(t) + XB(t)X + C(t) = 0$$

where $A$, $B$ and $C$ are functions from $J$ into $\mathcal{B}(H)$.

THEOREM 1.7.1 *Suppose* $J = [a,b]$ *and* $A, B, C \in C[J,\mathcal{B}(H)]$. *Then the*

IVP

$$X' + A^*(t)X + XA(t) + XB(t)X + C(t) = 0$$
$$X(t_o) = X_o \tag{1}$$

*has a unique local solution for every* $(t_o, X_o) \in J \times \mathcal{B}(H)$.

Proof. This theorem follows directly from the Picard-Lindelof theorem (Theorem 1.6.1). Let

$$f(t,X) = -[A^*(t)X+XA(t)+XB(t)X+C(t)]. \tag{2}$$

Clearly, for each fixed $X \in \mathcal{B}(H)$, $f(t,X)$ is continuous for all $t \in J$. If $t_o \in (a,b)$, pick $\alpha, \beta > 0$ such that $R = B(t_o;\alpha) \times B(X_o;\beta) \subset J \times \mathcal{B}(H)$. Let $K_1 = \sup_t \|A(t)\|$, $K_2 = \sup_t \|B(t)\|$ and $K_3 = \sup_t \|C(t)\|$, where the supremum is taken over $B(t_o;\alpha)$. We have

$$\|f(t,X)\| = \|A^*(t)X+XA(t)+XB(t)X+C(t)\|$$
$$\leq 2K_1(\|X_o\|+\beta) + K_2(\|X_o\|+\beta)^2 + K_3$$

for all $(t,X) \in R$. Also,

$$\|f(t,X)-f(t,Y)\|$$
$$= \|[A^*(t)X+XA(t)+XB(t)X+C(t)] - [A^*(t)Y+YA(t)+YB(t)Y+C(t)]\|$$
$$= \|A^*(t)(X-Y) + (X-Y)A(t) + XB(t)(X-Y) + (X-Y)B(t)Y\|$$
$$\leq [2K_1+2(\|X_o\|+\beta)]\|X-Y\|$$

for all $(t,X), (t,Y) \in R$. Thus by Theorem 1.6.1 (with $\mathcal{B}(H)$ as our B-space) the IVP (1) has a unique local solution. Now if $t_o = a$ the above statements still hold if we take $R = [t_o, t_o+\alpha] \times B(X_o;B)$ and we

conclude that (1) has a unique solution to the right. Similarly, if $t_o$ is the right endpoint of $J$ then (1) has a unique solution to the left.

□

This next theorem explains why we have chosen the scalar field for $H$ to be $\mathcal{R}$.

<u>THEOREM 1.7.2</u> *If* $A \in C[J,\mathcal{B}(H)]$ *and* $B, C \in C[J,\mathcal{S}]$, *where* $J = [a,b]$, *then the* IVP (1) *has a unique local <u>self-adjoint</u> solution for every* $(t_o,X_o) \in J \times \mathcal{S}$.

<u>Proof</u>. If $(t,X) \in J \times \mathcal{S}$ then

$$f(t,X)^* = -[A^*(t)X+XA(t)+XB(t)X+C(t)]^*$$

$$= -[XA(t)+A^*(t)X+XB(t)X+C(t)]$$

$$= f(t,X).$$

Thus $f(t,X)$ is a function from $J \times \mathcal{S}$ into $\mathcal{S}$. That $f(t,X)$ is continuous in $t$ and Lipschizian in $X$ is proved just as in the previous theorem. Thus, by Theorem 1.6.1 (with $\mathcal{S}$ as our B-space now), (1) has a unique local self-adjoint solution.

□

Theorem 1.7.2 holds only if $H$ is a <u>real</u> Hilbert space. Note that under the conditions of this theorem a solution $X(t)$ of (1) is self adjoint on some interval $J_1$ iff $X(t_o)$ is self-adjoint for some $t_o \in J_1$.

We now consider the behaviour of a noncontinuable solution of (1).

<u>THEOREM 1.7.3</u> *Suppose that* $X_o \in \mathcal{B}(H)$, *and* $A, B, C \in C[[t_o,b),\mathcal{B}(H)]$. *If* $X(t)$ *is a solution of* (1) *such that its right maximal interval of existence is* $[t_o,a)$, *with* $a < b$, *then* $\|X(t)\| \to \infty$ *as* $t \to a^-$.

<u>Proof</u>. Let $f(t,X)$ be as in equation (2). Clearly $f \in C[[t_o,b)\times\mathcal{B}(H), \mathcal{B}(H)]$. In the course of proving Theorem 1.7.1 we showed that $f$ maps bounded sets into bounded sets. Also, by Theorem 1.7.1, the IVP (1) has a local solution to the right for each $(t_o,X_o) \in [t_o,b) \times \mathcal{B}(H)$. The conditions of Theorem 1.6.3 are therefore met and this theorem follows immediately from it. □

Finally we show that solutions of (1) depend "continuously" on the initial data.

<u>THEOREM 1.7.4</u> *Suppose* $J$ *is a real interval and* $A, B, C \in C[J,\mathcal{B}(H)]$. *Let* $t_o \in [a,b] \subset J$ *and suppose* $X(t)$ *is a solution of the* IVP (1) *whose domain contains* $[a,b]$. *Suppose that* $(t_n,X_n,P_n) \to (t_o,X_o,0)$ *in* $J \times \mathcal{B}(H) \times \mathcal{B}(H)$ *as* $n \to \infty$. *If* $U_n(t)$ *is the noncontinuable solution of the* IVP

$$U_n' + A^*(t)U_n + U_nA(t) + U_nB(t)U_n + C(t) + P_n = 0$$

$$U_n(t_n) = X_n \tag{3}$$

*for* $n = 1,2,3,\ldots$, *then there exists a subsequence* $\{U_{n_i}(t)\}$ *such that* $U_{n_i}(t) \to X(t)$ *uniformly on* $[a,b]$.

<u>Proof</u>. This theorem follows from Corollary 1.6.5. We have to show conditions (i) through (iv) of Theorem 1.6.4 hold:

(i) We have

$$f_n(t,X) = -[A^*(t)X+XA(t)+XB(t)X+C(t)+P_n]$$

for $n = 1,2,3,\ldots$ and $f_o(t,X) = f(t,X)$. Clearly $f_n \in C[J\times\mathcal{B}(H),\mathcal{B}(H)]$ for $n = 0,1,2,\ldots$ .

(ii) The solution $X(t)$ is unique by Theorem 1.7.1.

(iii) If $\|X\|$, $\|Y\| \leq R$ then

$$\|f_n(t,X)-f_n(t,Y)\|$$

$$= \|A^*(t)(X-Y)+(X-Y)A(t)+XB(t)(X-Y)+(X-Y)B(t)Y\|$$

$\leq \rho_R(t)\|X-Y\|$ for all $(t,X)$, $(t,Y) \in J \times \mathcal{B}(H)$ where

$$\rho_R(t) = 2\|A(t)\| + 2R\|B(t)\| .$$

Thus condition (iii) of Theorem 1.6.4 is satisfied.

(iv) Clearly, $f_n(t,X) \to f(t,X)$ uniformly on $J \times \mathcal{B}(H)$.

The conditions of Theorem 1.6.4 are met and hence this theorem has been proved. □

## §1.8 Linear Differential Equations - Fundamental Solutions

If $B(t) \equiv 0$ in the Riccati equation we get a linear differential equation. Linear differential equations in Banach algebras have been dealt with by, for example, E. Hille [13] and most of the results of this section can be found in his text.

Consider the following initial value problem for a linear differential equation:

$$X' = A(t) + B(t)X + XC(t), \quad X(t_o) = X_o \tag{1}$$

All theorems from section 1.7 also apply to this initial value problem. For example, by Theorem 1.7.1, (1) has a unique local solution for every $(t_o, X_o) \varepsilon J \times \mathcal{B}(H)$. However, a much stronger global result can be proved:

THEOREM 1.8.1 *Let* $J = [a,b]$, $t_o \varepsilon J$, *and suppose that* $A, B, C \varepsilon C[J, \mathcal{B}(H)]$. *Then the* IVP (1) *has a unique solution on for every* $(t_o, X_o) \varepsilon J \times \mathcal{B}(H)$.

Proof. We proceed by the classical method of successive approximations: Let

$$X_o(t) = X_o$$

and

$$X_n(t) = X_o + \int_{t_o}^{t} [A(s)+B(s)X_{n-1}(s)+X_{n-1}(s)C(s)]ds$$

for $n \geq 1$. By induction, it is easily seen that each $X_n(t)$ is differentiable on $J$. Let

$$K_1 = \sup_{t\varepsilon J} \| B(t) \| + \sup_{t\varepsilon J} \| C(t) \|$$

and

$$K_2 = \int_a^b \| A(s) \| ds.$$

Then for $n \geq 2$ and all $t \varepsilon J$ we have

$$\| X_n(t)-X_{n-1}(t) \|$$

$$= \| \int_{t_o}^{t} [B(s)(X_{n-1}(s)-X_{n-2}(s))+(X_{n-1}(s)-X_{n-2}(s))C(s)]ds \|$$

$$< K_1 \int_{t_o}^{t} \| X_{n-1}(s)-X_{n-2}(s) \| ds \tag{2}$$

We first prove the following assertion:

<u>Assertion</u>: $\| X_n(t)-X_{n-1}(t) \| \leq K_2K_1^{n-1} \dfrac{|t-t_o|^{n-1}}{(n-1)!} + \|X_o\| K_1^n \dfrac{|t-t_o|^n}{n!}$

for all $t \varepsilon J$ and $n \geq 1$.

We prove the assertion by induction. The claim is true for $n = 1$ since

$$\| X_1(t)-X_o(t) \| = \| \int_{t_o}^{t} [A(s)+B(s)X_o+X_oC(s)]ds \|$$

$$\leq K_2 + \| X_o \| K_1 |t-t_o| .$$

Using (2) we now get

$$\| X_{n+1}(t)-X_n(t) \|$$

$$\leq K_1 \int_{t_o}^{t} \left[ K_2K_1^{n-1} \frac{|s-t_o|^{n-1}}{(n-1)!} + \| X_o \| K_1^n \frac{|s-t_o|^n}{n!} \right] ds$$

$$= K_2K_1^n \frac{|t-t_o|^n}{n!} + \| X_o \| K_1^{n+1} \frac{|t-t_o|^{n+1}}{(n+1)!}$$

thereby proving our assertion.

The sequence $\{X_n(t)\}$ thus converges uniformly on $J$ to a necessarily continuous limit, say $X(t)$. $X(t)$ satisfies the equation

$$X(t) = X_o + \int_{t_o}^{t} [A(s)+B(s)X(s)+X(s)C(s)]ds$$

and hence is a solution of (1) on J.

Now suppose that $Y(t)$ is also a solution of (1) in a neighbourhood of $t = t_o$. Then

$$\begin{aligned} \|X(t)-Y(t)\| &= \Big\| \int_{t_o}^{t} [B(s)(X(s)-Y(s)) \\ &\quad + (X(s)-Y(s))C(s)]ds \Big\| \\ &\le K_1 \int_{t_o}^{t} \|X(s)-Y(s)\| ds. \end{aligned}$$

Applying Gronwall's lemma to the non-negative function $h(t) = \|X(t)-Y(t)\|$ we see that $h(t) \equiv 0$ and hence $X(t) = Y(t)$ wherever $Y(t)$ exists. □

<u>COROLLARY 1.8.2</u> *Theorem* 1.8.1 *holds on any interval* J.

<u>Proof</u>. Let $a$ and $b$ be the endpoints of $J$ with $a < b$. The corollary follows immediately upon applying the theorem to intervals of the form $[a_n,b_n]$ where $a_n$ $(b_n)$ approaches $a$ $(b)$ from the right (left). □

Now let $J$ be an open interval, $(t_o,X_o) \in J \times \mathcal{B}(H)$, and $A \in C[J,\mathcal{B}(H)]$. Let $X(t;t_o,X_o)$ denote the unique solution on $J$ of the IVP

$$X' = A(t)X, \quad X(t_o) = X_o.$$

We will say $X(t;t_o,X_o)$ is a <u>fundamental solution</u> if $X_o$ is non-singular. The main result of this section is to show that a fundamental solution is non-singular throughout $J$. But first we prove the following lemma:

<u>LEMMA 1.8.3</u> *Let* $J = (a,b)$, $A \in C[J,\mathcal{B}(H)]$ *and* $J_1 = [a_1,b_1] \subset J$. *Then for any* $\varepsilon > 0$ *and any* $\alpha \in J_1$ *there exists a* $\delta > 0$ *independent of* $\alpha$ *such that* $\| X(t;\alpha,I)-I\| < \varepsilon$ *for all* $|t-\alpha| < \delta$.

<u>Proof</u>. Let $J_2 = (a_2,b_2)$ and $J_3 = [a_3,b_3]$ be intervals such that $a < a_3 < a_2 < a_1$ and $b > b_3 > b_2 > b_1$. Then $J_1 \subset J_2 \subset J_3 \subset J$. For $n = 1,2,3,\ldots,$ let

$$S_n \equiv \left\{ \lambda \in J_2 \;\middle|\; \begin{array}{l} \| X(t;\lambda,I)-I\| < \varepsilon \\ \text{for all } \; t \in J \cap B(\lambda;\, \frac{1}{n}) \end{array} \right\}$$

where $B(\lambda; \frac{1}{n})$ is the closed interval of radius $\frac{1}{n}$ and center $\lambda$. The theorem will be proved if we can show that $S_{n_o} \supset J_1$ for some $n_o$.

Clearly,

$$S_n \subset S_{n+1} \quad \text{for} \quad n = 1,2,3,\ldots \tag{3}$$

and

$$\bigcup_1^\infty S_n = J_2.$$

Let $K$ be a positive integer such that $K > \max\left\{\dfrac{1}{a_2-a_3}, \dfrac{1}{b_3-b_2}\right\}$.

<u>Claim</u>: $S_n$ is open for all $n \geq K$.

<u>Proof of Claim</u>. Let $\lambda \varepsilon S_k$ for some $k \geq K$. Note that $B(\lambda; \frac{1}{k}) \subset J_3$ by our choice of $K$. In fact, there exists a $\delta_1 > 0$ such that $B(\lambda; \frac{1}{k} + \delta_1) \subset J_3$ also. Since $\lambda \varepsilon S_k$ then $\|X(t;\lambda,I)-I\| < \varepsilon$ whenever $t \varepsilon B(\lambda; \frac{1}{k})$. But by the continuity of $X(t;\lambda,I)$ there exists a $\delta_2 > 0$ such that we also have $\|X(t;\lambda,I)-I\| < \varepsilon$ whenever $t \varepsilon B(\lambda; \frac{1}{k} + \delta_2)$. Let $c_1 = \sup\limits_t \|X(t;\lambda,I)-I\|$, where the supremum is taken over $B(\lambda; \frac{1}{k} + \delta_2)$. Note that $c_1 < \varepsilon$. Let $c_2 = \varepsilon - c_1 > 0$. Now let $\{\lambda_n\}$ be a sequence of numbers converging to $\lambda$. Pick $N_1$ so that $|\lambda_n - \lambda| < \min\{\delta_1, \delta_2\}$ for all $n \geq N_1$. By Theorem 1.7.4, there exists a subsequence $\{\lambda_{n_i}\}$ such that $X(t;\lambda_{n_i},I)$ converges uniformly on $J_3$ to $X(t;\lambda,I)$. Pick $N_2$ so that $\|X(t;\lambda_{n_i},I) - X(t;\lambda,I)\| < c_2$ for all $n_i \geq N_2$ and all $t \varepsilon J_3$. Now suppose that $n_i \geq \max\{N_1,N_2\}$ and $t \varepsilon B(\lambda_{n_i}; \frac{1}{k})$. Then $t \varepsilon J_3 \cap B(\lambda; \frac{1}{k} + \delta_2)$ and hence we obtain

$$\begin{aligned} &\|X(t;\lambda_{n_i},I)-I\| \\ &\leq \|X(t;\lambda_{n_i},I)-X(t;\lambda,I)\| \\ &+ \|X(t;\lambda,I)-I\| < c_2 + c_1 = \varepsilon \end{aligned}$$

for all $t \varepsilon B(\lambda_{n_i}; \frac{1}{k})$. Thus $\lambda_{n_i} \varepsilon S_k$ for all $n_i > \max\{N_1,N_2\}$. This proves $S_k$ is open for all $k \geq K$ and establishes our claim.

$\{S_n\}_K^\infty$ is therefore an open covering of the compact set $J_1$ and hence, in view of (3), there exists a positive integer $N$ such that $S_n \supseteq J_1$ for all $n \geq N$. Therefore the theorem holds with $\delta = \frac{1}{N}$. $\square$

We now prove the main result of this section.

THEOREM 1.8.4 *If* $J = (a,b)$, $A \in C[J,\mathcal{B}(H)]$, *and* $t_o \in J$ *then* $X(t;t_o,I)$ *is non-singular throughout* $J$.

Proof. Since $X(t;t_o,I)$ is non-singular at $t = t_o$ it is also non-singular in a neighbourhood of $t_o$ (by Theorem 1.3.3). Suppose that $X(t;t_o,I)$ fails to be non-singular for all $t \in J$. Then there exists a neighbourhood $(t_o-h_1,t_o+h)$ of $t_o$ such that $X(t;t_o,I)$ is non-singular throughout this neighbourhood but is singular at an endpoint, say at $t = t_o + h \in J$.

For any $\alpha \in [t_o,t_o+h]$ it is possible, by Lemma 1.8.3, to pick a $\delta > 0$ independent of $\alpha$ such that $\|X(t;\lambda,I)-I\| < \frac{1}{2}$ for all $t \in B(\alpha;2\delta)$. Then, by Corollary 1.3.2, $X(t;\alpha,I)$ is non-singular for all $t \in B(\alpha;2\delta)$. Let $\alpha = t_o + h - \delta$. Note that $X(t;t_o,I)$ and $X(t;t_o+h-\delta,I)X(t_o+h-\delta;t_o,I)$ are both solutions of $X' = A(t)X$ that are equal when $t = t_o + h - \delta$. By uniqueness of solutions (Corollary 1.8.2) we therefore have

$$X(t;t_o,I) = X(t;t_o+h-\delta,I)X(t_o+h-\delta;t_o,I) \tag{4}$$

for all $t \in J$. Note that $X(t;t_o+h-\delta,I)$ is non-singular for $t \in [t_o+h-3\delta,t_o+h+\delta]$ and $X(t_o+h-\delta;t_o,I)$ is non-singular. Therefore, from (4), we see that $X(t;t_o,I)$ is also non-singular for all $t \in [t_o+h-3\delta,t_o+h+\delta]$ contradicting the fact it is singular at $t = t_o + h$.

Therefore $X(t;t_o,I)$ is non-singular throughout $J$. □

<u>COROLLARY 1.8.5</u> *Under the same hypotheses as in Theorem* 1.8.4, $X(t;t_o,X_o)$ *is non-singular throughout* J *iff* $X_o$ *is non-singular.*

<u>Proof</u>. The proof follows immediately from Theorem 1.8.4 since

$$X(t;t_o,X_o) = X(t;t_o,I)X_o.$$

□

CHAPTER II

# COMPARISON THEOREMS FOR SELF-ADJOINT, $C^*$-VALUED RICCATI DIFFERENTIAL EQUATIONS

## §2.1 Properties of Solutions of Differential Equations and Differential Inequalities of Riccati Type

Let $J$ be a real interval, $H$ a real Hilbert space and $\mathcal{S}$ the set of self-adjoint operators in the $C^*$-algebra $\mathcal{B}(H)$. Consider the self-adjoint, $C^*$-valued Riccati differential equation

$$R[X](t) \equiv X'(t) + A^*(t)X(t) + X(t)A(t) + X(t)B(t)X(t) + C(t) = 0 \tag{1}$$

where $A$, $B$ and $C$ are functions from $J$ into $\mathcal{S}$. If $H = \mathcal{R}^n$ then (1) is a self-adjoint matrix Riccati equation. In this chapter we generalize to the $C^*$-algebra case the basic properties of self-adjoint matrix Riccati equations that are to be found in the texts of W.A. Coppel ([2]) and W.T. Reid ([23]).

The following assumption will hold throughout this section:

Assumption: $A,B,C \in C[J,\mathcal{S}]$

In this section we shall study the properties of solutions of both the Riccati differential equation $R[X](t) = 0$ and of the Riccati differential inequalities obtained when the equality is replaced by some inequality. We first study the relationship between two solutions of (1).

THEOREM 2.1.1 *Let* $J = (a,b)$. *If* $X_1(t)$ *and* $X_2(t)$ *are self-adjoint solutions of* (1) *on* $J$ *such that* $X_2(c) \geq X_1(c)$ $[X_2(c) > X_1(c)]$

*for some* $c \in J$ *then* $X_2(t) \geq X_1(t)$ $[X_2(t) > X_1(t)]$ *throughout* $J$.

<u>Proof</u>. Let $W(t) = X_2(t) - X_1(t)$ and $Y(t) = \frac{X_1(t)+X_2(t)}{2}$. It is easily verified that $W(t)$ solves the differential equation

$$\begin{aligned} W' &+ W[A(t)+B(t)Y(t)] \\ &+ [A^*(t)+Y(t)B(t)]W = 0 \end{aligned}$$

on $J$. Also, if $U(t)$ is the solution of

$$U' = [A(t)+B(t)Y(t)]U, \quad U(c) = I$$

then we have

$$\begin{aligned} [U^*(t)W(t)U(t)]' &= U'^*WU + U^*W'U + U^*WU' \\ &= U^*(A+BY)^*WU + U^*[-W(A+BY) \\ &- (A^*+YB)W]U + U^*W(A+BY)U \\ &= 0. \end{aligned}$$

Since $U^*(c)W(c)U(c) = W(c)$ and $U(t)$ exists and is non-singular throughout $J$ (by Theorem 1.8.4) it follows that

$$W(t) = U^{*-1}(t)W(c)U^{-1}(t)$$

for all $t \in J$. Therefore, if $W(c) \geq 0$ $[> 0]$ then $W(t) \geq 0$ $[> 0]$ throughout $J$. That is, if $X_2(c) \geq X_1(c)$ $[X_2(c) > X_1(c)]$ then $X_2(t) \geq X_1(t)$ $[X_2(t) > X_1(t)]$ throughout $J$. □

Now suppose that $X(t)$ is a solution of (1) but that $Y(t)$ only satisfies a Riccati differential inequality. A result similar to

Theorem 2.1.1 still holds but a condition on the spectrum of the initial values of $X(t)$ and $Y(t)$ has to be added.

<u>THEOREM 2.1.2</u> *Suppose that on* $J = [a,b]$ *(or* $[a,b)$*)* $X(t)$ *is a self-adjoint solution of* $R[X](t) = 0$ *and* $Y(t)$ *is a self-adjoint solution of the differential inequality* $R[Y](t) > 0$ $[< 0]$ *such that* $Y(a) > X(a)$ $[Y(a) < X(a)]$ *and* $0 \notin \sigma[Y(a)-X(a)]$. *Then* $Y(t) > X(t)$ $[Y(t) < X(t)]$ *throughout* $J$.

<u>Proof</u>. Suppose that $R[Y](t) > 0$ on $J$ and $Y(a) > X(a)$. Since $0 \notin \sigma[Y(a)-X(a)]$ then $Y(t) > X(t)$ for all $t$ in some right neighbourhood of $t = a$ (by Corollary 1.5.9). Suppose that the inequality $Y(t) > X(t)$ fails to hold throughout $J$. Then there exists a number $c \in (a,b]$ such that $Y(t) > X(t)$ on $[a,c)$ and $Y(c) \not> X(c)$. It follows from Corollary 1.5.7 that $Y(c) \geq X(c)$ and there exists a $x_o \in H$, $x_o \neq 0$ such that $Y(c)x_o = X(c)x_o$. Define $g:J \to \mathcal{R}$ by

$$g(t) = ([Y(t)-X(t)]x_o,x_o),$$

Clearly $g(t) > 0$ on $[a,c)$ and $g(c) = 0$. However, we have

$$\begin{aligned}
0 &< (R[Y](c)x_o,x_o) \\
&= ([R[Y](c)-R[X](c)]x_o,x_o) \\
&= ([Y'(c)-X'(c)+A^*(c)(Y(c)-X(c)) \\
&\qquad +(Y(c)-X(c))A(c)+Y(c)B(c)Y(c) \\
&\qquad -X(c)B(c)X(c)]x_o,x_o) \\
&= ([Y'(c)-X'(c)]x_o,x_o) \\
&\qquad + (A^*(c)(Y(c)-X(c))x_o,x_o)
\end{aligned}$$

$$+ (A(c)x_o, (Y(c)-X(c))x_o)$$

$$+ ([Y(c)B(c)Y(c)-X(c)B(c)X(c)]x_o, x_o)$$

$$= ([Y'(c)-X'(c)]x_o, x_o)$$

$$+ (Y(c)B(c)[Y(c)-X(c)]x_o, x_o)$$

$$+ (B(c)X(c)x_o, [Y(c)-X(c)]x_o)$$

$$= ([Y'(c)-X'(c)]x_o, x_o)$$

$$= g'(c).$$

Thus $g'(c) > 0$ which is a contradiction. Thus the inequality $Y(t) > X(t)$ must hold throughout $J$.

The other part of this theorem is similarly proved. □

By means of a limit argument and using the fact that the solution of a Riccati initial value problem depends continuously on the initial data we obtain the following stronger version of Theorem 2.1.2 where the condition $0 \notin \sigma[Y(a)-X(a)]$ has been eliminated and the condition that $R[Y](t) > 0$ on $J$ has been weakened to $R[Y](t) \geq 0$ on $J$.

<u>THEOREM 2.1.3</u> *Suppose that on* $J = [a,b]$ *(or* $[a,b)$*)* $X(t)$ *is a self-adjoint solution of* $R[X]t = 0$, *and* $Y(t)$ *is a self-adjoint solution of the inequality* $R[Y](t) \geq 0$ $[\leq 0]$ *such that* $Y(a) \geq X(a)$ $[Y(a) \leq X(a)]$. *Then* $Y(t) \geq X(t)$ $[Y(t) \leq X(t)]$ *throughout* $J$.

<u>Proof</u>. Suppose that $R[Y](t) \geq 0$ throughout $J$ and $Y(a) \geq X(a)$. For each $k = 1,2,3,\ldots,$ let $X_k(t)$ be the self-adjoint solution of the Riccati equation

$$\tilde{R}_k[X_k](t) \equiv R[X_k](t) + \frac{1}{k} I = 0$$

satisfying the initial condition

$$X_k(a) = X(a) - \frac{1}{k} I.$$

Let $a < c \in J$. Then, by Theorem 1.7.4, there exists a subsequence $\{X_{k_i}(t)\}_i$ such that $X_{k_i}(t) \to X(t)$ uniformly on $[a,c]$. Note that $\tilde{R}_{k_i}[Y](t) > 0$ on $[a,c]$ and $Y(a) > X_{k_i}(a)$ for $i = 1,2,3,\ldots$ . Also, since

$$\sigma[Y(a)-X_{k_i}(a)] = \sigma[Y(a)-X(a)] + \frac{1}{k_i}$$

and

$$\sigma[Y(a)-X(a)] \subset [0,\infty)$$

we see that $0 \notin \sigma[Y(a)-X_{k_i}(a)]$ for $i = 1,2,3,\ldots$ . Hence, by Theorem 2.1.2, $Y(t) > X_{k_i}(t)$ on $[a,c]$ for $i = 1,2,3,\ldots$ and therefore, by Theorem 1.5.2, $Y(t) \geq X(t)$ on $[a,c]$. Since $c$ was arbitrary in $J$, we must have $Y(t) \geq X(t)$ throughout $J$. The other part of this corollary is similarly proved. □

Our next result is a global existence theorem. It shows that if $X(t)$ is a solution of (1) such that $Y(a) \geq X(a) \geq Z(a)$ where $Y(t)$ and $Z(t)$ satisfy certain differential inequalities then $X(t)$ exists and is "bounded" by $Y(t)$ and $Z(t)$ throughout $J$.

<u>COROLLARY 2.1.4</u> *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $Y(t)$ *and* $Z(t)$ *are self-adjoint solutions of the inequalities*

$R[Y](t) \geq 0$ *and* $R[Z](t) \leq 0$, *respectively, on* $J$.

(ii) $Y(a) > Z(a)$

(iii) $D$ *is a self-adjoint operator such that* $Y(a) \geq D \geq Z(a)$.

*Then the self-adjoint solution* $X(t)$ *of* $R[X](t) = 0$ *satisfying* $X(a) = D$ *exists and satisfies* $Y(t) \geq X(t) \geq Z(t)$ *throughout* $J$.

Proof. Let $[a,\alpha)$ be the right maximal interval of existence of $X(t)$. By Theorem 2.1.3, the inequality $Y(t) \geq X(t) \geq Z(t)$ holds on $[a,\alpha)$. By Theorem 1.5.5, $X(t)$ is bounded on $[a,\alpha)$. We must therefore have $\alpha = b$, by Theorem 1.7.3. Thus the inequality $Y(t) \geq X(t) \geq Z(t)$ holds on $[a,b)$ and hence throughout $J$. □

We now show that Theorem 2.1.3 holds even if the inequality signs are replaced by strict inequality signs.

THEOREM 2.1.5 *Suppose that* $J = [a,b)$ *and the following hold:*

(i) $X(t)$ *is a self-adjoint solution of* $R[X](t) = 0$ *on* $J$.

(ii) $Y(t)$ *is a self-adjoint solution of the inequality* $R[Y](t) \geq 0$ $[\leq 0]$ *on* $J$ *such that* $Y(a) > X(a)$ $[Y(a) < X(a)]$.

*Then* $Y(t) > X(t)$ $[Y(t) < X(t)]$ *throughout* $J$.

Proof. Suppose that $R[Y](t) \geq 0$ on $J$ and $Y(a) > X(a)$. Let $Z(t)$ be the self-adjoint solution of $R[Z](t) = 0$ that satisfies $Z(a) = Y(a)$. Since $R[Y](t) \geq 0$ and $R[X](t) \leq 0$ on $J$ then, by Corollary 2.1.4, $Z(t)$ exists and satisfies $Y(t) \geq Z(t) \geq X(t)$ throughout $J$. By Theorem 2.1.1 however we must have $Z(t) > X(t)$ throughout $J$. Thus $Y(t) > X(t)$ throughout $J$. The other part of this corollary is

similarly proved. □

A corollary to Theorem 2.1.5 is the following result which shows that Corollary 2.1.4 holds with the inequality signs replaced by strict inequality signs.

<u>COROLLARY 2.1.6</u> *Suppose that* $J = [a,b)$ *and condition* (i) *of Corollary* 2.1.4 *holds. In addition, if* $Y(a) > Z(a)$ *and* $D$ *is a self-adjoint operator such that* $Y(a) > D > Z(a)$ *then the self-adjoint solution* $X(t)$ *of* $R[X](t) = 0$ *satisfying* $X(a) = D$ *is defined and satisfies* $Y(t) > X(t) > Z(t)$ *throughout* $J$.

<u>Proof</u>. Let $[a,\alpha)$ be the right maximal interval of existence of $X(t)$. By Theorem 2.15, the inequality $Y(t) > X(t) > Z(t)$ holds on $[a,\alpha)$. By Theorem 1.5.5, $X(t)$ is bounded on $[a,\alpha)$ and hence, by Theorem 1.7.3, we must have $\alpha = b$. Thus $Y(t) > X(t) > Z(t)$ throughout $J = [a,b)$. □

If $B(t)$ is non-negative or non-positive throughout $J$ then the hypotheses of Corollary 2.1.4 and Corollary 2.1.6 can be weakened.

<u>THEOREM 2.1.7</u> *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $B(t) \geq 0$ $[\leq 0]$ *on* $J$.

(ii) $Y(t)$ *is a self-adjoint solution of the inequality* $R[Y](t) \leq 0$ $[\geq 0]$ *on* $J$.

*Then any self-adjoint solution of* $R[X](t) = 0$ *that satisfies* $X(a) \geq Y(a)$ $[X(a) \leq Y(a)]$ *is defined and satisfies* $X(t) \geq Y(t)$

$[X(t) \leq Y(t)]$ *throughout* $J$.

Proof. Suppose that $B(t) \geq 0$ and $R[Y](t) \leq 0$ on $J$ and that $X(a) \geq Y(a)$. Let $Z(t)$ be the self-adjoint solution of the linear differential equation

$$\begin{aligned} \tilde{R}[Z](t) \equiv Z' + ZA(t) + A(t)Z \\ + C(t) - I = 0 \end{aligned}$$

that satisfies

$$Z(a) = X(a) + I.$$

$Z(t)$ exists throughout $J$ by Theorem 1.8.1. Also, for all $t \in J$,

$$\begin{aligned} R[Z](t) &= \tilde{R}[Z](t) + Z(t)B(t)Z(t) + I \\ &= Z(t)B(t)Z(t) + I \\ &> 0 \end{aligned}$$

where the last inequality follows from the fact that $I > 0$ and $Z(t)B(t)Z(t) \geq 0$ on $J$. Since $Z(a) > X(a) > Y(a)$ then, by Corollary 2.1.4, $X(t)$ is defined and satisfies $Z(t) \geq X(t) \geq Y(t)$ throughout $J$.

The other part of this corollary is similarly proved. □

THEOREM 2.1.8 *Suppose that* $J = [a,b)$ *and conditions* (i) *and* (ii) *of Theorem* 2.1.7 *hold. Then any self-adjoint solution* $X(t)$ *of* $R[X](t) = 0$ *that satisfies* $X(a) > Y(a)$ $[X(a) < Y(a)]$ *is defined and satisfies* $X(t) > Y(t)$ $[X(t) < Y(t)]$ *throughout* $J$.

Proof. Suppose $B(t) \leq 0$ and $R[Y](t) \geq 0$ on $J$ and $X(a) < Y(a)$. Let $Z(t)$ be the self-adjoint solution of the linear differential

equation

$$\tilde{R}[Z](t) \equiv Z' + ZA(t) + A(t)Z + C(t) + I$$
$$= 0$$

that satisfies $Z(a) = X(a) - I$. $Z(t)$ exists throughout $J$ by Theorem 1.8.1. Also, for all $t \in J$, we have

$$\begin{aligned} R[Z](t) &= \tilde{R}[Z](t) + Z(t)B(t)Z(t) - I \\ &= Z(t)B(t)Z(t) - I \\ &< 0 \end{aligned}$$

where the inequality follows from that fact that $-I < 0$ and $Z(t)B(t)Z(t) \leq 0$ on $J$. Since $Y(a) > X(a) > Z(a)$ then, by Corollary 2.1.6, $X(t)$ is defined and satisfies $Y(t) > X(t) > Z(t)$ throughout $J$.

The other part of this theorem is similarly proved. □

Suppose that $X(t)$ is a solution of (1) such that $X(a) \geq 0$ or $X(a) \leq 0$. The following results give sufficient conditions for $X(t)$ to exist and be non-negative or non-positive throughout $J$.

COROLLARY 2.1.9 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $B(t) \geq 0$ $[\leq 0]$ *on* $J$.

(ii) $C(t) \leq 0$ $[\geq 0]$ *on* $J$.

*Then any self-adjoint solution* $X(t)$ *of* $R[X](t) = 0$ *that satisfies* $X(a) \geq 0$ $[\leq 0]$ *is defined and satisfies* $X(t) \geq 0$ $[\leq 0]$ *throughout* $J$.

Proof. This follows directly from Theorem 2.1.7 if we take $Y(t) \equiv 0$ on $J$. □

COROLLARY 2.1.10 *Suppose that* $J = [a,b]$ *and conditions* (i) *and* (ii) *of Corollary 2.1.9 hold. Then any self-adjoint solution* $X(t)$ *of* $R[X](t) = 0$ *that satisfies* $X(a) > 0$ $[< 0]$ *is defined and satisfies* $X(t) > 0$ $[< 0]$ *throughout* $J$.

Proof. This follows directly from Theorem 2.1.8 if we take $Y(t) \equiv 0$ on $J$. □

## §2.2 Standard Comparison Theorems

Now consider the following pair of Riccati differential equations:

$$R_i[X](t) \equiv X' + A_i(t)X + XA_i(t) + XB_i(t)X + C_i(t) = 0, \quad i = 1,2$$

With the aid of the results from the previous section we have ready proofs of the following comparison theorems.

THEOREM 2.2.1 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $A_i(t)$, $B_i(t)$ *and* $C_i(t)$ *belong to* $C[J,\mathcal{S}]$ *and* $i = 1,2$.

(ii) *For each* $t \in J$, $(A_2(t)-A_1(t))$ $[A_1(t)-A_2(t)]$ *is a non-negative scalar operator. (A scalar operator is an element of the closed subspace of* $\mathcal{B}(H)$ *that is generated by the identity operator* $I$*).*

(iii) $B_2(t) \geq B_1(t) \geq 0$ $[B_2(t) \leq B_1(t) \leq 0]$ *on* $J$.

(iv) $0 \geq C_2(t) \geq C_1(t)$ $[0 \leq C_2(t) \leq C_1(t)]$ *on* $J$.

(v) $X_2(t)$ *is a self-adjoint solution of* $R_2[X](t) = 0$ *on* $J$ *such that* $X_2(a) \geq 0$ $[\leq 0]$.

*Then any self-adjoint solution* $X_1(t)$ *of* $R_1[X](t) = 0$ *such that* $X_1(a) \geq X_2(a)$ $[X_1(a) \leq X_2(a)]$ *is defined and satisfies* $X_1(t) \geq X_2(t)$ $[X_1(t) \leq X_2(t)]$ *throughout* $J$.

<u>Proof</u>. Suppose that the first set of hypotheses hold. Since $X_2(a) \geq 0$ then, by Corollary 2.1.9, $X_2(t) \geq 0$ throughout $J$. Also, for all $t \in J$,

$$\begin{aligned} & R_2[X_2](t) - R_1[X_2](t) \\ &= [A_2(t)-A_1(t)]X_2(t) + X_2(t)[A_2(t)-A_1(t)] \\ &+ X_2(t)[B_2(t)-B_1(t)]X_2(t) + [C_2(t)-C_1(t)] \\ &\geq 0 \end{aligned}$$

where the inequality follows from conditions (ii), (iii) and (iv). Thus $R_1[X_2](t) \leq R_2[X_2](t) = 0$ on $J$. By Theorem 2.1.7 therefore, $X_1(t)$ is defined and satisfies $X_1(t) \geq X_2(t)$ throughout $J$.

The other part of the theorem is similarly proved. □

<u>COROLLARY 2.2.2</u> *Suppose that* $J = [a,b)$ *and conditions* (i) *through* (v) *of Theorem 2.2.1 hold. Then any self-adjoint solution* $X_1(t)$ *of* $R_1[X](t) = 0$ *such that* $X_1(a) > X_2(a)$ $[X_1(a) < X_2(a)]$ *is defined and satisfied* $X_1(t) > X_2(t)$ $[X_1(t) < X_2(t)]$ *throughout* $J$.

<u>Proof</u>. Suppose that the first set of hypotheses hold. As in Theorem 2.2.1, we again have $R_1[X_2](t) \leq 0$ on $J$. Since $X_1(a) > X_2(a)$ then,

by Theorem 2.1.8, we must have $X_1(t) > X_2(t)$ throughout $J$.

The other part of the corollary is proved similarly. □

If $A_1(t) = A_2(t)$ on $J$ then we obtain the following stronger versions of Theorem 2.2.1 and Corollary 2.2.2.

THEOREM 2.2.3 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $A_i(t)$, $B_i(t)$ *and* $C_i(t)$ *belong to* $C[J,\mathcal{S}]$ *for* $i = 1,2$ *and* $A_1(t) = A_2(t) \equiv A(t)$ *on* $J$.

(ii) $B_2(t) \geq B_1(t) \geq 0$ $[B_2(t) \leq B_1(t) \leq 0]$ *on* $J$.

(iii) $C_2(t) - C_1(t) \geq 0$ $[\leq 0]$ *on* $J$.

(iv) $X_2(t)$ *is a self-adjoint solution of the inequality* $R_2[X](t) \leq 0$ $[\geq 0]$ *throughout* $J$.

*Then any self-adjoint solution* $X_1(t)$ *of* $R_1[X](t) = 0$ *that satisfies* $X_1(a) \geq X_2(a)$ $[X_1(a) \leq X_2(a)]$ *is defined and satisfies* $X_1(t) \geq X_2(t)$ $[X_1(t) \leq X_2(t)]$ *throughout* $J$.

Proof. Suppose that the first set of hypotheses hold. Then for all $t \in J$,

$$\begin{aligned} &R_2[X_2](t) - R_1[X_2](t) \\ &= X_2(t)[B_2(t)-B_1(t)]X_2(t) + [C_2(t)-C_1(t)] \\ &\geq 0 \end{aligned}$$

where the inequality follows from conditions (ii) and (iii). Thus $R_1[X_2](t) \leq R_2[X_2](t) \leq 0$ on $J$. By Theorem 2.1.7 therefore, $X_1(t)$

is defined and satisfies $X_1(t) \geq X_2(t)$ throughout $J$.

The other part of the theorem is proved similarly. $\square$

<u>COROLLARY 2.2.4</u> *Suppose that* $J = [a,b)$ *and conditions* (i) *through* (iv) *of Theorem* 2.2.3 *hold. Then any self-adjoint solution* $X_1(t)$ *of* $R_1[X](t) = 0$ *such that* $X_1(a) > X_2(a)$ $[X_1(a) < X_2(a)]$ *is defined and satisfies* $X_1(t) > X_2(t)$ $[X_1(t) < X_2(t)]$ *throughout* $J$.

<u>Proof</u>. Suppose that the first set of hypotheses hold. As in Theorem 2.2.3, we again have $R_1[X_2](t) \leq 0$ throughout $J$. Thus, by Theorem 2.1.8, $X_1(t)$ is defined and satisfies $X_1(t) > X_2(t)$ throughout $J$.

The other part of the corollary is proved similarly. $\square$

CHAPTER III

COMPARISON THEOREMS OF INTEGRAL TYPE

In this chapter we give comparison theorems for the following pair of Riccati equations:

$$X' + X^2 + Q_1(t) = 0 \tag{1}$$

$$Y' + Y^2 + Q(t) = 0 \tag{2}$$

where $Q_1$ and $Q$ are functions from some real interval $J$ into $\mathcal{S}$. The comparison theorems in the previous chapter were those in which the coefficients of the differential equations (i.e. $A_i(t)$, $B_i(t)$ and $C_i(t)$) satisfied some simple inequalities. In the comparison theorems of this chapter, it will be the integrals of $Q_1(t)$ and $Q(t)$ that satisfy an inequality.

It will be necessary for us to assume that $Q_1(t)$ is a scalar operator and that (1) has a scalar operator solution for all $t \in J$. The following assumptions will hold throughout this chapter:

<u>Assumptions:</u>

(1) $Q_1(t) = q_1(t)I$ and $Q(t)$ belong to $C[J,\mathcal{S}]$.

(2) The Riccati differential equation (1) has a scalar operator solution $X(t) = x(t)I$ on $J$.

The comparison theorems in this chapter are generalizations of comparison theorems given by R.A. Jones ([14]) for matrix Riccati equations. In attempting to generalize these theorems, an immediate

difficulty arises. The proofs for the matrix case make use of the fact that if $A(t)$ is continuous in $t$ and $A(t_o) > 0$ then $A(t) > 0$ for all $t$ in some neighbourhood of $t_o$. This of course is not true if $H$ is infinite dimensional unless we have the additional assumption that $0 \notin \sigma(A(t_o))$.

In this next theorem we overcome this difficulty by first proving an assertion with the additional condition that $0 \notin \sigma(A(t_o))$ and then using a limit argument similar to that used in the proof of Theorem 2.1.3 to eliminate this additional condition.

<u>THEOREM 3.1</u> *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $\mu$ *is an element of* $C[J,(0,\infty)]$ *such that*

$$\mu'(t)I \geq \mu(t)X(t)$$

*and*

$$2\mu''(t)I + \frac{2}{\mu(t)}[\mu'(t)I-\mu(t)X(t)]^2 + \mu(t)[Q_1(t)+Q(t)] \geq 0$$

*on* $J$.

(ii) $\int_a^t \mu^2(s)Q_1(s)ds \geq \int_a^t \mu^2(s)Q(s)ds$

*for all* $t \in J$.

*Then* (2) *has a self-adjoint solution* $Y(t)$ *defined and satisfying*

$$X(t) \le Y(t) \le \frac{2\mu'(t)}{\mu(t)} I - X(t)$$

*throughout* $J$.

Proof. Under the transformations

$$U(t) = \mu(t)X(t)$$

and

$$V(t) = \mu(t)Y(t),$$

equations (1) and (2) become

$$\mu(t)U' = \mu'(t)U - U^2 - \mu^2(t)Q_1(t) \tag{3}$$

and

$$\mu(t)V' = \mu'(t)V - V^2 - \mu^2(t)Q(t), \tag{4}$$

respectively. Note that by assumption (2) and hypothesis (i), $U(t) = \mu(t)x(t)I$ is a scalar operator solution of (3) satisfying

$$U(t) \le \mu'(t)I \tag{5}$$

and

$$2\mu''(t) + \frac{2}{\mu(t)} [\mu'(t)I-U(t)]^2$$
$$+ \mu(t)[Q_1(t)+Q(t)] \ge 0 \tag{6}$$

on $J$.

To prove the theorem, it suffices to show the existence of a self-adjoint solution $V(t)$ of (4) satisfying

$$U(t) \leq V(t) \leq 2\mu'(t)I - U(t)$$

throughout $J$.

We now consider the following assertion:

Assertion: If instead of (6) we have the stronger condition

$$2\mu''(t) + \frac{2}{\mu(t)}[\mu'(t)I-U(t)]^2$$
$$+ \mu(t)[Q_1(t)+Q(t)] > 0 \quad \text{on} \quad J \tag{7}$$

and in addition we have

$$U(a) < \mu'(a)I \tag{8}$$

then any solution $V(t)$ of (4) satisfying

$$U(a) < V(a) < \mu'(a)I, \tag{9}$$
$$0 \notin \sigma[2\mu'(a)I-U(a)-V(a)] \quad \text{and}$$
$$0 \notin \sigma[V(a)-U(a)]$$

exists and satisfies

$$U(t) \leq V(t) \leq 2\mu'(t)I - U(t)$$

throughout $J$.

Proof of assertion: Let $[a,\alpha)$ be the right maximal interval of existence of $V(t)$. From (9) we have $V(a) < 2\mu'(a)I - U(a)$. Suppose the inequality $V(t) < 2\mu'(t)I - U(t)$ fails to hold throughout $[a,\alpha)$. Since $0 \notin \sigma[2\mu'(a)I-U(a)-V(a)]$ then, by Corollary 1.5.9, there exists a $c \in (a,\alpha)$ such that $2\mu'(t)I - U(t) - V(t) > 0$ on $(a,c)$ and

$2\mu'(c)I - U(c) - V(c) \not> 0$. Thus, by Corollary 1.5.7, there exists a $x_o \in H$, $x_o \neq 0$ such that $[2\mu'(c)I-U(c)-V(c)]x_o = 0$. Now define $g: [a,\alpha) \to \mathcal{R}$ by

$$g(t) = ([2\mu'(t)I-U(t)-V(t)]x_o,x_o).$$

Then $g(t)$ is differentiable on $[a,\alpha)$, $g(t) > 0$ on $[a,c)$ and $g(c) = 0$.

Since $V(t)$ is a solution of (4) on $[a,\alpha)$ we have,

$$\begin{aligned}(V'(t)x_o,x_o) &= \frac{1}{\mu(t)}([\mu'(t)V(t)-V^2(t)]x_o,x_o)\\ &\quad - \mu(t)(Q(t)x_o,x_o)\end{aligned}$$

on $[a,\alpha)$. Let $M(t) \equiv \mu'(t)I - U(t)$. Then $V(c)x_o = [\mu'(c)I+M(c)]x_o$ and hence

$$\begin{aligned}(V'(c)x_o,x_o) &= \frac{1}{\mu(c)}([\mu'(c)(\mu'(c)I+M(c))\\ &\quad - (\mu'(c)I+M(c))^2]x_o,x_o) - \mu(c)(Q(c)x_o,x_o)\\ &= -\frac{1}{\mu(c)}(\mu'(c)M(c)x_o,x_o)\\ &\quad - \frac{1}{\mu(c)}(M^2(c)x_o,x_o) - \mu(c)(Q(c)x_o,x_o)\\ &< -\frac{1}{\mu(c)}(\mu'(c)M(c)x_o,x_o)\\ &\quad + 2\mu''(c)(x_o,x_o) + \frac{1}{\mu(c)}(M^2(c)x_o,x_o)\\ &\quad + \mu(c)(Q_1(c)x_o,x_o) && \text{by (7)}\\ &= ([\mu''(c)I+M'(c)]x_o,x_o) && \text{by (3).}\end{aligned}$$

Therefore

$$g'(c) = ([\mu''(c)I+M'(c)-V'(c)]x_o,x_o)$$

$$> 0,$$

contradicting the fact that $g(t) > 0$ on $[a,c)$ and $g(c) = 0$. Thus

$$V(t) < 2\mu'(t)I - U(t) \quad \text{for all} \quad t \in [a,\alpha).$$

Now suppose that the inequality $U(t) < V(t)$ fails to hold throughout $[a,\alpha)$. Since $V(a) > U(a)$ and $0 \notin \sigma[V(a)-U(a)]$ then, by Corollary 1.5.9, there exists a $c \in (a,\alpha)$ such that $U(c) < V(t)$ on $[a,c)$ and $U(c) \nless V(c)$. Thus, as before, there exists a $x_o \in H$, $x_o \neq 0$ such that $U(c)x_o = V(c)x_o$. From (3) we obtain

$$\begin{aligned}[\mu(t)U(t)]' &= \mu(t)U'(t) + \mu'(t)U(t) \\ &= 2\mu'(t)U(t) - U^2(t) - \mu^2(t)Q_1(t) \quad \text{on} \quad J.\end{aligned}$$

Integrating this equation over $(a,t)$ gives us

$$\begin{aligned}U(t) = \frac{\mu(a)}{\mu(t)} U(a) &+ \frac{1}{\mu(t)} \int_a^t [2\mu'(s)U(s)-U^2(s)]ds \\ &- \frac{1}{\mu(t)} \int_a^t \mu^2(s)Q_1(s)ds\end{aligned}$$

for all $t \in [a,\alpha)$. Similarly, from (4) we obtain

$$\begin{aligned}V(t) = \frac{\mu(a)}{\mu(t)} V(a) &+ \frac{1}{\mu(t)} \int_a^t [2\mu'(s)V(s)-V^2(s)]ds \\ &- \frac{1}{\mu(t)} \int_a^t \mu^2(s)Q_1(s)ds\end{aligned}$$

for all $t \in [a,\alpha)$. On $[a,\alpha)$, we therefore have

$$V(t) - U(t) = \frac{\mu(a)}{\mu(t)}[V(a)-U(a)]$$

$$+ \frac{1}{\mu(t)}\int_a^t [(2\mu'(s)V(s)-V^2(s))$$

$$- (2\mu'(s)U(s)-U^2(s))]ds$$

$$+ \frac{1}{\mu(t)}\int_a^t \mu^2(s)[Q_1(s)-Q(s)]ds.$$

At $t = c$ we therefore have

$$0 = \frac{\mu(a)}{\mu(c)}([V(a)-U(a)]x_o,x_o)$$

$$+ \frac{1}{\mu(c)}(\int_a^c [(2\mu'(s)V(s)-V^2(s))$$

$$- (2\mu'(s)U(s)-U^2(s))]ds\ x_o,x_o)$$

$$+ \frac{1}{\mu(c)}(\int_a^c \mu^2(s)[Q_1(s)-Q(s)]ds\ x_o,x_o). \tag{10}$$

For $t \in [a,c)$ we have

$$U(t) < V(t) < 2\mu'(t)I - U(t).$$

Hence $0 < [\mu'(t)I-U(t)] + [V(t)-\mu'(t)I]$ and $0 < [\mu'(t)I-U(t)] - [V(t)-\mu'(t)I]$. The right hand sides of the above inequalities commute (since $U(t)$ is a scalar operator) and hence, by Theorem 1.5.10, we get

$$0 < [\mu'(t)I-U(t)]^2 - [V(t)-\mu'(t)I]^2$$

$$= [2\mu'(t)V(t)-V^2(t)] - [2\mu'(t)U(t)-U^2(t)]$$

for all $t \in [a,c)$. Therefore, the first two terms on the right in

equation (10) are positive and the last is non-negative, which is a contradiction. We must therefore have

$$U(t) < V(t) < 2\mu'(t)I - U(t)$$

throughout $[a,\alpha)$. Then, by Theorem 1.5.5, $V(t)$ is bounded on $[a,\alpha)$. Thus $Y(t)$ is also bounded on $[a,\alpha)$, where $Y(t) \equiv \frac{V(t)}{\mu(t)}$ is a solution of (2). Therefore, by Theorem 1.7.3, we must have $\alpha = b$. Thus the inequality

$$U(t) < V(t) < 2\mu'(t)I - U(t)$$

holds on $[a,b)$ and hence the inequality

$$U(t) \leq V(t) \leq 2\mu'(t)I - U(t)$$

holds throughout $J$. This proves our assertion.

Suppose now that the original hypothesis of the theorem holds. By (5) we have $\mu'(a)I \geq U(a)$. For any real number $\lambda$, define the self-adjoint operator $A_\lambda$ by $A_\lambda = (1-\lambda)U(a) + \lambda\mu'(a)I$. Then

$$\sigma[A_\lambda] = \{\lambda[\mu'(a)-\mu(a)x(a)]+\mu(a)x(a)\}.$$

Suppose now that $\mu'(a) \neq \mu(a)x(a)$. Then for any real number $c$, there exists exactly one value of $\lambda$ for which $c \in \sigma[A_\lambda]$. Thus the set of all numbers $\lambda$ for which

$$\mu(a)x(a) - \frac{1}{k} \in \sigma[A_\lambda] \tag{11}$$

or

$$2\mu'(a) - \mu(a)x(a) + \frac{3}{2k} \in \sigma[A_\lambda] \tag{12}$$

for some positive integer $k$ is countable. Therefore we can pick a number $\lambda_o \in (0,1)$ such that when $\lambda = \lambda_o$ neither (11) nor (12) holds for any positive integer $k$.

Now suppose that $\mu'(a) = \mu(a)x(a)$. Then $\sigma[A_{\lambda_o}] = \{\mu(a)x(a)\}$. Thus again neither (11) nor (12) holds for any positive integer $k$ when $\lambda = \lambda_o$.

Now let $V(t)$ denote the self-adjoint solution of (4) satisfying $V(a) = A_{\lambda_o}$. Let $[a,\alpha)$ be the right maximal interval of existence of $V(t)$. Note that by our choice of $\lambda_o$, we have $U(a) \leq V(a) \leq 2\mu'(a)I - U(a)$. We must have $U(t) \leq V(t) \leq 2\mu'(a)I - U(t)$ throughout $[a,\alpha)$, for suppose not. Let $c \in [a,\alpha)$ be the largest number such that the inequality $U(t) \leq V(t) \leq 2\mu'(a)I - U(t)$ holds on $[a,c]$. For each $k = 1,2,3,\ldots$, let $U_k(t)$ $[V_k(t)]$ be the solution of (3) [(4)] satisfying $U_k(a) = U(a) - \frac{1}{k} I$ $[V_k(a) = V(a) - \frac{1}{2k} I]$. Note that for any $k = 1,2,3,\ldots$, we have $U(a) - \frac{1}{k} I < V(a) - \frac{1}{2k} I < \mu'(a)I$ and hence $U_k(a) < V_k(a) < \mu'(a)I$. Note also that

$$\sigma[V_k(a)-U_k(a)] = \sigma[A_{\lambda_o}] - (\mu(a)x(a) - \frac{1}{2k})$$

and

$$\sigma[2\mu'(a)I-U_k(a)-V_k(a)] = (2\mu'(a)-\mu(a)x(a)+\frac{3}{2k}) - \sigma[A_{\lambda_o}]$$

and thus by our choice of $\lambda_o$ we have

$$0 \notin \sigma[V_k(a)-U_k(a)]$$

and

$$0 \notin \sigma[2\mu'(a)I-U_k(a)-V_k(a)]$$

for all positive integers $k$.

Now let $\delta \varepsilon (0,\alpha-c)$. By Theorem 1.7.4, there exists a subsequence $\{U_{k_i}(t)\}$ $[\{V_{k_i}(t)\}]$ that converges uniformly on $[a,c+\delta]$ to $U(t)$ $[V(t)]$. Also, by Theorem 2.1.5, we have $\frac{U_{k_i}(t)}{\mu(t)} < \frac{U(t)}{\mu(t)}$ on $[a,c+\delta]$. Thus for each $i = 1,2,3,\ldots,$ we have

$$0 \leq \mu'(t)I - U(t) < \mu'(t)I - U_{k_i}(t)$$

on $[a,c+\delta)$ and hence

$$2\mu''(t)I + \frac{2}{\mu(t)}[\mu'(t)I-U_{k_i}(t)]^2$$

$$+ \mu(t)[Q_1(t)+Q(t)] > 0 \quad \text{on} \quad [a,c+\delta].$$

By our assertion, we therefore have

$$U_{k_i}(t) \leq V_{k_i}(t) \leq 2\mu'(t)I - U_{k_i}(t)$$

on $[a,c+\delta]$ for each $i = 1,2,3,\ldots$ . Since the set of non-negative operators is closed in $\mathcal{B}(H)$ we therefore have

$$U(t) \leq V(t) \leq 2\mu'(t)I - U(t) \quad \text{on} \quad [a,c+\delta]$$

contradicting our choice of $c$. Thus

$$U(t) \leq V(t) \leq 2\mu'(t)I - U(t) \quad \text{on} \quad [a,\alpha).$$

By Theorem 1.5.5, $V(t)$ is bounded on $[a,\alpha)$ and hence, by Theorem 1.7.3, we have $\alpha = b$. Thus $Y(t) = \frac{V(t)}{\mu(t)}$ is a self-adjoint solution of (2) which exists and satisfies

$$X(t) \leq Y(t) \leq \frac{2\mu'(t)}{\mu(t)} I - X(t)$$

throughout $J$. □

A result similar to Theorem 3.1 also exists for solutions to the left.

<u>COROLLARY 3.2</u> *Suppose that* $J = [a,b]$ *(or* $(a,b]$*) and the following hold:*

(i) $\mu$ *is an element of* $C[J,(0,\infty)]$ *such that*

$$\mu'(t)I \leq \mu(t)X(t) \tag{13}$$

*and*

$$2\mu''(t)I + \frac{2}{\mu(t)} [\mu'(t)I-\mu(t)X(t)]^2 + \mu(t)[Q_1(t)+Q(t)] \geq 0 \tag{14}$$

*on* $J$.

(ii) $$\int_t^b \mu^2(s)Q_1(s) \geq \int_t^b \mu^2(s)Q(s)\,ds \tag{15}$$

*on* $J$.

*Then* (2) *has a self-adjoint solution* $Y(t)$ *defined and satisfying*

$$\frac{2\mu'(t)}{\mu(t)} I - X(t) \le Y(t) \le X(t)$$

*throughout* J.

Proof. Let $c \in J$. Then the function $g(t) = c + b - t$ maps $[c,b]$ onto itself. We make the following transformations:

$$U(t) = -X(g(t)), \quad V(t) = -Y(g(t))$$

$$R_1(t) = Q_1(g(t)), \quad R(t) = Q(g(t))$$

and $\rho(t) = \mu(g(t))$.

Then $U(t)$ is a scalar operator solution of

$$U' + U^2 + R_1(t) = 0$$

on $[c,b]$ and (2) becomes

$$V' + V^2 + R(t) = 0. \tag{16}$$

From (13) we obtain

$$\rho(t)U(t) \le \rho'(t)I$$

on $[c,b]$. Making the change of variable $s = g(u)$, we obtain from (15),

$$\int_c^{g(t)} \rho^2(u)R_1(u)\,du \ge \int_c^{g(t)} \rho^2(u)R(u)\,du$$

for all $t \in [c,b]$. Or equivalently,

$$\int_c^t \rho^2(u)R_1(u)\,du \geq \int_c^t \rho^2(u)R(u)\,du$$

for all $t \in [c,b]$. Replacing $t$ by $g(t)$ in (14), we get

$$2\rho''(t)I + \frac{2}{\rho(t)}[\rho'(t)I-\rho(t)U(t)]^2$$

$$+ \rho(t)[R_1(t)+R(t)] \geq 0$$

for all $t \in [c,b]$. Thus, by Theorem 3.1, (16) has a self-adjoint solution, $V(t)$, defined and satisfying

$$U(t) \leq V(t) \leq \frac{2\rho'(t)}{\rho(t)} I - U(t)$$

throughout $[c,b]$. Or equivalently,

$$-X(g(t)) \leq -Y(g(t)) \leq \frac{-2\mu'(g(t))}{\mu(g(t))} + X(g(t))$$

for all $t \in [c,b]$, where $Y(t)$ is a self-adjoint solution of (2) defined by $Y(t) = -V(g(t))$ for all $t \in [c,b]$. From the last inequality we also obtain

$$\frac{2\mu'(t)}{\mu(t)} I - X(t) \leq Y(t) \leq X(t)$$

for all $t \in [c,b]$. Since this is true for arbitrary $c$ in $J$ then $Y(t)$ exists and satisfies

$$\frac{2\mu'(t)}{\mu(t)} I - X(t) \leq Y(t) \leq X(t)$$

throughout $J$. $\square$

In the following comparison theorem, limit arguments such as the one used in the proof of Theorem 3.1 do not work. We therefore have an additional condition on the spectrum of the initial value, a condition that is not needed for the proof of the matrix case.

THEOREM 3.3 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and* $A$ *is a self-adjoint bounded operator such that*

$$\begin{aligned} -X(a) + \int_a^t Q_1(s)ds &\geq -A + \int_a^t Q(s)ds \\ &\geq X(a) - \int_a^t Q_1(s)ds \end{aligned} \tag{17}$$

*for all* $t \in J$ *and*

$$\{-x(a), x(a)\} \cap \sigma[A] = \phi. \tag{18}$$

*Then the self-adjoint solution* $Y(t)$ *of* (2) *satisfying* $Y(a) = A$ *exists and satisfies*

$$X(t) \leq Y(t) \leq -X(t)$$

*throughout* $J$.

Proof. We first consider the following claim:

Claim: $Y(t)$ satisfies the conclusion of the theorem if $A$ satisfies the stronger condition

$$\begin{aligned} -X(a) + \int_a^t Q_1(s)ds &> -A + \int_a^t Q(s)ds \\ &> X(a) - \int_a^t Q_1(s)ds \end{aligned} \tag{19}$$

for all $t \in J$.

Proof of claim: Let $[a,\alpha)$ be the right maximal interval of existence of $Y(t)$. Putting $t = a$ in (19) gives us $-X(a) > -Y(a) > X(a)$ or $X(a) < Y(a) < -X(a)$. Suppose that the inequality $Y(t) < -X(t)$ fails to hold throughout $[a,\alpha)$. We have $\sigma[-X(a)-Y(a)] = -x(a) - \sigma[Y(a)]$ and hence, by (18), $0 \notin \sigma[-X(a)-Y(a)]$. Thus, by Corollary 1.5.9, there exists a number $c \in (a,\alpha)$ such that $Y(t) < -X(t)$ on $[a,c)$ and $Y(c) \nless -X(c)$. By Corollary 1.5.7, there then exists a $x_o \in H$, $x_o \neq 0$, such that $Y(c)x_o = -X(c)x_o$. For all $t \in [a,\alpha)$ we have

$$-Y(t) = -Y(a) + \int_a^t Y^2(s)ds + \int_a^t Q(s)ds$$

and hence, using (19), we get

$$\begin{aligned} -Y(c) &= -Y(a) + \int_a^c Y^2(s)ds + \int_a^c Q(s)ds \\ &> X(a) - \int_a^c Q_1(s)ds - \int_a^c X^2(s)ds \\ &= X(c). \end{aligned}$$

This contradicts the fact that

$$([-Y(c)-X(c)]x_o,x_o) = 0.$$

Therefore $Y(t) < -X(t)$ on $[a,\alpha)$.

Now suppose that the inequality $X(t) < Y(t)$ fails to hold throughout $[a,\alpha)$. We have $\sigma[Y(a)-X(a)] = \sigma[Y(a)] - x(a)$ and hence,

by (18), $0 \notin \sigma[Y(a)-X(a)]$. Thus, by Corollary 1.5.9., there exists a number $c \in (a,\alpha)$ such that $Y(t) > X(t)$ on $[a,c)$ and $Y(c) \not> X(c)$. By Corollary 1.5.7., there exists a $x_o \in H$, $x_o \neq 0$ such that $X(c)x_o = Y(c)x_o$. We now have $X(t) < Y(t) < -X(t)$ on $[a,c)$ and since $X(t)$ and $Y(t)$ commute we also have

$$\begin{aligned} 0 &< [Y(t)-X(t)][-X(t)-Y(t)] \\ &= X^2(t) - Y^2(t) \end{aligned}$$

on $[a,c)$. Using (19), we thus get

$$\begin{aligned} -Y(c) &= -Y(a) + \int_a^c Q(s)ds + \int_a^c Y^2(s)ds \\ &< -X(a) + \int_a^c Q_1(s)ds + \int_a^c X^2(s)ds \\ &= -X(c). \end{aligned}$$

This contradicts the fact that $([-X(c)+Y(c)]x_o,x_o) = 0$. Thus $X(t) < Y(t) < -X(t)$ on $[a,\alpha)$. Then, by Theorem 1.5.5, $Y(t)$ is bounded on $[a,\alpha)$ and, by Theorem 1.7.3, we must have $\alpha = b$. Thus $X(t) < Y(t) < -X(t)$ on $[a,b)$ and hence $X(t) \leq Y(t) \leq -X(t)$ throughout $J$. This proves our claim.

Now suppose that $A$ satisfies the hypothesis of the theorem. Let $[a,\alpha)$ be the right maximal interval of existence of $Y(t)$. Putting $t = a$ in (17) we see that $X(a) \leq Y(a) \leq -X(a)$. Suppose the inequality $X(t) \leq Y(t) \leq -X(t)$ fails to hold throughout $[a,\alpha)$. Let $c \in [a,\alpha)$ be the largest number such that the inequality $X(t) \leq Y(t) \leq -X(t)$ holds on $[a,c]$. Note that for each $k = 1,2,3,\ldots$ ,

$$(-X(a)+\frac{1}{k}I)+\int_a^t Q_1(s)ds > -Y(a)$$

$$+\int_a^t Q(s)ds > (X(a)-\frac{1}{k}I)-\int_a^t Q_1(s)ds$$

for all $t \in J$. For $k = 1,2,3,\ldots$, let $X_k(t)$ be the scalar operator solution of (1) such that $X_k(a) = X(a) - \frac{1}{k}I$. Let $\delta \in (0,\alpha-c)$. By Theorem 1.7.4, there exists a subsequence $\{X_{k_i}(t)\}$ which converges uniformly to $X(t)$ on $[a,c+\delta]$. Also, since $x(a)$ and $-x(a)$ are elements of the open set $\rho[Y(a)]$ then there exists a $N > 0$ such that

$$\{x(a)-\frac{1}{k},-x(a)+\frac{1}{k}\} \cap \sigma[Y(a)] = \phi$$

for all $k \geq N$. By our claim therefore

$$X_{k_i}(t) \leq Y(t) \leq -X_{k_i}(t) \quad \text{on} \quad [a,c+\delta]$$

for all $k_i \geq N$. Hence $X(t) \leq Y(t) \leq -X(t)$ on $[a,c+\delta]$ contradicting the definition of $c$. Thus the inequality

$$X(t) \leq Y(t) \leq -X(t)$$

holds for all $t \in [a,\alpha)$. By Theorem 1.5.5, $Y(t)$ is bounded on $[a,\alpha)$ and hence, by Theorem 1.7.3, $\alpha = b$. Therefore, the inequality

$$X(t) \leq Y(t) \leq -X(t)$$

holds throughout $J$. This completes the proof of the theorem. □

A result similar to Theorem 3.3 also exists for solutions to the left.

COROLLARY 3.4 *Suppose that* $J = [a,b]$ *(or* $(a,b]$*) and* $A$ *is a self-adjoint bounded operator such that*

$$X(b) + \int_t^b Q_1(s)\,ds \geq -A + \int_t^b Q(s)\,ds$$

$$\geq -X(b) - \int_t^b Q_1(s)\,ds \quad \text{on} \quad J \tag{20}$$

*and*

$$\{-x(b), x(b)\} \cap \sigma[A] = \phi. \tag{21}$$

*Then the self-adjoint solution* $Y(t)$ *of* (2) *satisfying* $Y(b) = -A$ *exists and satisfies*

$$-X(t) \leq Y(t) \leq X(t)$$

*throughout* $J$.

Proof. Let $c \in J$. Then the function $g(t) = c + b - t$ maps $[c,b]$ onto itself. We make the following transformations:

$$U(t) = -X(g(t)), \quad V(t) = -Y(g(t))$$

$$R(t) = Q(g(t)), \quad R_1(t) = Q_1(g(t)).$$

Then $U(t)$ is a scalar operator solution of

$$U' + U^2 + R_1(t) = 0 \quad \text{on} \quad [c,b]$$

and (2) becomes

$$V' + V^2 + R(t) = 0. \tag{22}$$

Noting that $X(b) = -U(c)$ and making the change of variables $s = g(u)$, (3) becomes

$$-U(c) + \int_c^{g(t)} R_1(u)\,du \geq -A + \int_c^{g(t)} R(u)\,du$$

$$\geq U(c) - \int_c^{g(t)} R_1(u)\,du \quad \text{for all} \quad t \in [c,b].$$

Or equivalently,

$$-U(c) + \int_c^{t} R_1(u)\,du \geq -A + \int_c^{t} R(u)\,du$$

$$\geq U(c) - \int_c^{t} R_1(u)\,du \quad \text{for all} \quad t \in [c,b].$$

Note also that $\sigma[-U(c)] = \{x(b)\}$ and hence by (21),

$$\sigma[-U(c)] \cap \sigma[A] = \phi \quad \text{and} \quad \sigma[U(c)] \cap \sigma[A] = \phi.$$

Therefore, by Theorem 3.3, there exists a solution $V(t)$ of (22) that exists and satisfies $U(t) \leq V(t) \leq -U(t)$ throughout $[c,b]$. Then $Y(t) \equiv -V(g(t))$ is a solution of (2) that exists and satisfies $-X(t) \leq Y(t) \leq X(t)$ throughout $[c,b]$. Since this is true for arbitrary $c$ in $J$, then $Y(t)$ exists and satisfies $-X(t) \leq Y(t) \leq X(t)$ throughout $J$. Note that $Y(t)$ is the solution of (2) for which $Y(b) = -V(c) = -A$. $\square$

# CHAPTER IV

## APPLICATIONS AND ADDITIONAL COMPARISON THEOREMS

In this chapter we establish a form of Hille's comparison theorem. Our result is a generalization to the $C^*$-algebra case of a comparison theorem that Heidel [12] gives for real scalar functions. We also combine the results of the previous two chapters to formulate additional comparison theorems for the following Riccati differential equations:

$$R_1[X](t) \equiv X' + \lambda(t)X + XP(t)X + Q(t) = 0 \tag{1}$$

$$R_2[X](t) \equiv X' + XP(t)X + Q(t) = 0 \tag{2}$$

$$R_3[X](t) \equiv X' + X^2 + Q(t) = 0. \tag{3}$$

$P$ and $Q$ are functions from some real interval $J$ into $\mathcal{S}$, and $\lambda$ is a real-valued function on $J$. Unless otherwise specified the following three assumptions hold throughout this chapter:

<u>Assumptions</u>:

(1) $P \in C[J,\mathcal{S}]$ and $0 \leq P(t) \leq I$ on $J$.

(2) $Q \in C[J,\mathcal{S}]$.

(3) $\lambda \in C[J,(-\infty,0]]$.

The following theorem gives sufficient conditions for (1) to have a positive solution on $J$.

<u>THEOREM 4.1</u> *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $Q_1 \in C[J,S]$ *and* $Q_1(t) \geq Q(t)$ *on* $J$.

(ii) $X(t)$ *is a positive solution of the inequality* $X' + X^2 + Q_1(t) \leq 0$ *on* $J$.

*Then any self-adjoint solution* $Y(t)$ *of* (1) *that satisfies* $Y(a) \geq X(a)$ *exists and satisfies* $Y(t) \geq X(t) > 0$ *throughout* $J$.

Proof. Let $Z(t)$ be the self-adjoint solution of (2) such that $Z(a) = X(a)$. By assumption (1) we have $0 \leq P(t) \leq I$ on $J$ and hence, by Theorem 2.2.3, $Z(t)$ exists and satisfies $Z(t) \geq X(t)$ on $J$. However,

$$R_1[Z](t) = \lambda(t)Z(t) \leq 0$$

on $J$. Thus, by Theorem 2.1.7, $Y(t)$ exists and satisfies $Y(t) \geq Z(t) \geq X(t) > 0$ throughout $J$. □

Note that $X(t) = (\frac{1}{2t})I$ is a positive scalar operator solution of $X' + X^2 + (\frac{1}{4t^2})I = 0$ on any positive real interval. An application of Theorem 4.1 is the following:

COROLLARY 4.2 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*), with* $a > 0$, *and* $Q(t) \leq (\frac{1}{4t^2})I$ *on* $J$. *Then any self-adjoint solution* $Y(t)$ *of* (1) *that satisfies* $Y(a) \geq (\frac{1}{2a})I$ *exists and satisfies* $Y(t) \geq (\frac{1}{2t})I$ *throughout* $J$.

Proof. The proof follows immediately from Theorem 4.1 if we take $Q_1(t) = (\frac{1}{4t^2})I$ and $X(t) = (\frac{1}{2t})I$. □

The following theorem gives sufficient conditions for (2) to

have negative solutions.

THEOREM 4.3 *Suppose that* $Q(t) \geq (\frac{1}{4}+\varepsilon)t^{-2}I$ *and* $P(t) \geq I$ *on* $J = [a,\infty)$, *where* $\varepsilon$ *and* $a$ *are positive real numbers. If* $X(t)$ *is a self-adjoint solution of* (2) *such that* $X(a) \leq 0$, *then* $X(t) < 0$ *throughout* $(a,\alpha)$, *the right maximal interval of existence of* $X(t)$.

Proof. Let $y(t)$ be the real valued solution of the differential equation

$$y' + y^2 + (\frac{1}{4}+\varepsilon)t^{-2} = 0 \tag{4}$$

that satisfies the initial condition $y(a) = M(X(a))$. Let $[a,\beta)$ be the right maximal interval of existence of $y$. It is a well known fact (e.g. Hartman [10], p. 362) that the second order differential equation corresponding to the Riccati equation (4) is oscillatory. Thus $\beta < \infty$ and it follows that $y(t) \downarrow -\infty$ as $t \to \beta^-$. Since $y(a) \leq 0$, we see that $y(t) < 0$ throughout $(a,\beta)$. The scalar operator $y(t)I$ satisfies the differential inequality

$$R_2[y(t)I](t) = y^2(t)[P(t)-I]$$

$$+ [Q(t)-(\frac{1}{4} + )t^{-2}I] \geq 0$$

for all $t \in [a,\beta)$. Since $X(a) \leq y(a)I$ then, by Theorem 2.1.3,

$X(t) \leq y(t)I < 0$ for all $t \in (a,\alpha) \cap (a,\beta)$. It follows that $\alpha \leq \beta$ and hence $X(t) < 0$ throughout $(a,\alpha)$.

□

We now combine Theorem 3.1 with the results of Chapter II to obtain the following comparison theorem for (1).

THEOREM 4.4 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*),* $a > 0$, *and the following hold:*

(i) $\mu \in C[J,(0,\infty)]$ *and* $\mu'(t) \geq \frac{\mu(t)}{2t}$ *on* $J$.

(ii) $2\mu''(t)I + \frac{2}{\mu(t)}[\mu'(t) - \frac{\mu(t)}{2t}]^2 I + \mu(t)[\frac{1}{4t^2}I + Q(t)] \geq 0$ *on* $J$.

(iii) $\int_a^t \frac{\mu^2(s)}{4s^2} I\,ds \geq \int_a^t \mu^2(s)Q(s)\,ds$ *for all* $t \in J$.

*Then* (1) *has a positive solution* $Y(t)$ *that exists and satisfies* $Y(t) \geq (\frac{1}{2t})$ *throughout* $J$.

Proof. Note that the hypotheses of Theorem 3.1 are satisfied with $Q_1(t) = (\frac{1}{4t^2})I$ and $X(t) = (\frac{1}{2t})I$ and hence (3) has a self-adjoint solution $V(t)$ that exists and satisfies

$$(\frac{1}{2t})I \leq V(t) \leq [\frac{2\mu'(t)}{\mu(t)} - \frac{1}{2t}]I$$

throughout $J$. Let $Y(t)$ be the self-adjoint solution of (1) such that $Y(a) = V(a)$. Then, by Theorem 4.1, $Y(t)$ exists and satisfies $Y(t) \geq V(t) \geq (\frac{1}{2t})I$ throughout $J$.

□

If we take the special case $\mu(t) = t^{\theta}$ in Theorem 4.4 we obtain the following corollary:

COROLLARY 4.5 *Suppose that* $J = [a,b]$ *(or* $[a,b))$, $a > 0$, *and* $\theta$ *is a real number not less than* $\frac{1}{2}$ *such that*

$$4(\theta - \tfrac{1}{4})(\theta - \tfrac{3}{4})I + t^2 Q(t) \geq 0$$

*and*

$$\int_a^t \frac{s^{2(\theta-1)}}{4} I\, ds \geq \int_a^t s^{2\theta} Q(s)\, ds$$

*on* $J$. *Then* (1) *has a positive solution* $X(t)$ *that exists and satisfies* $X(t) \geq (\frac{1}{2t})I$ *throughout* $J$.

Proof. The proof follows immediately from Theorem 4.4. □

Another special case of Theorem 4.4 is the following:

COROLLARY 4.6 *Suppose that* $J = [a,b]$ *(or* $[a,b))$, $a > 0$, *and the following hold:*

(i) $\mu \in C[J,(0,\infty)]$

(ii) $\mu'(t) \geq \frac{\mu(t)}{2t}$ *and* $\mu''(t) \geq 0$ *on* $J$.

(iii) $\int_a^t \frac{\mu^2(s)}{4s^2}\, ds \geq \int_a^t \mu^2(s) Q(s)\, ds$ *for all* $t \in J$.

(iv) $Q(t) \geq (-\frac{1}{4t^2})I$ *on* $J$.

*Then* (1) *has a positive solution* $X(t)$ *that exists and satisfies* $X(t) \geq (\frac{1}{2t})$ *throughout* $J$.

Proof. The proof follows immediately from Theorem 4.4. □

Combining Theorem 3.3 with the comparison theorems of Chapter II gives us the following result:

THEOREM 4.7 *Suppose that* $J = [a,b]$ *(or* $[a,b)$*) and the following hold:*

(i) $Q_1(t) = q_1(t)I \in C[J,S]$

(ii) $X(t) = x(t)I$ *is scalar operator solution of* $X' + X^2 + Q_1(t) = 0$ *on* $J$.

(iii) $A$ *is a self-adjoint operator such that*

$$-X(a) + \int_a^t Q_1(s)ds \geq -A + \int_a^t Q(s)ds$$

$$\geq X(a) - \int_a^t Q_1(s)ds \quad \textit{for all} \quad t \in J$$

*and*
$$\{-x(a),x(a)\} \cap \sigma[A] = \phi.$$

*Then* (2) *has a self-adjoint solution* $Y(t)$ *that exists and satisfies* $Y(t) \geq X(t)$ *throughout* $J$.

Proof. Let $Y(t)$ and $Z(t)$ be self-adjoint solutions of (2) and (3), respectively, that satisfy the initial conditions $Y(a) = Z(a) = A$. Then, by Theorem 3.3, $Z(t)$ exists and satisfies $X(t) \leq Z(t) \leq -X(t)$ throughout $J$. By assumption (1) we have $0 \leq P(t) \leq I$ on $J$ and hence, by Theorem 2.2.3, $Y(t)$ exists and satisfies $Y(t) \geq Z(t) \geq X(t)$ throughout $J$. □

An application of Corollary 3.4 when $Q_1(t) = (\frac{1}{4t^2})I$ and $X(t) = (\frac{1}{2t})I$ is the following:

<u>THEOREM 4.8</u> *Suppose that* $J = [a,b]$ *(or* $(a,b)$*),* $a > 0$*, and* $A$ *is a self-adjoint operator such that*

$$\frac{1}{2b} I + \frac{b-t}{4bt} I \geq A + \int_t^b Q(s)ds$$

$$\geq -\frac{1}{2b} I + \frac{b-t}{4bt} I \quad \text{for all} \quad t \in J$$

and

$$\{-\frac{1}{2b}, \frac{1}{2b}\} \cap \sigma[A] = \phi.$$

*Then* (2) *has a self-adjoint solution* $Y(t)$ *that exists and satisfies* $Y(t) \geq (-\frac{1}{2t})I$ *throughout* $J$.

<u>Proof</u>. By Corollary 3.4, the self-adjoint solution $Z(t)$ of (3) that satisfies $Z(b) = A$ exists and satisfies $(-\frac{1}{2t})I \leq Z(t) \leq (\frac{1}{2t})I$ on $J$. By Theorem 2.2.3 therefore, (2) has a self-adjoint solution $Y(t)$ that exists and satisfies $Y(t) \geq Z(t) \geq (-\frac{1}{2t})I$ throughout $J$. □

The following form of Hille's comparison theorem is a generalization of a comparison theorem that Heidel [12] gives for the scalar case.

<u>THEOREM 4.9</u> *Let* $J = [a,\infty)$ *and suppose the following hold:*

(i) $Q_1(t) = q_1(t)I$ *is a continuous function from* $J$ *into* $S$.

(ii) $X(t) = x(t)I$ *is a positive scalar operator solution of the Riccati differential equation* $X' + X^2 + Q_1(t) = 0$ *on* $J$.

(iii) *The integrals* $\int_t^\infty Q_1(s)ds$ *and* $\int_t^\infty Q(s)ds$ *exist and satisfy*

$$\int_t^\infty Q_1(s)ds \geq \int_t^\infty Q(s)ds \geq 0 \quad \textit{for each} \quad t \in J.$$

*Then* (1) *has a positive solution on* J.

<u>Proof</u>. Note that $x(t)$ is a real, positive solution of

$$x' + x^2 + q_1(t) = 0 \tag{5}$$

on $J$ and that $\int_t^\infty q_1(s)ds$ exists for all $t \in J$. Therefore, by an exercise of Hartman ([10]), exercise 7.5, p. 365), we have $\lim_{t\to\infty} x(t) = 0$. Note also that $\lim_{T\to\infty} \frac{1}{T} \int_a^T \int_a^t q_1(s)ds\, dt = c$ exists and therefore, by a lemma of Hartman ([10], lemma 7.1, p. 365), we also have $\int_a^\infty x^2(t)dt < \infty$. Now suppose that $a \leq t < \lambda$. Integrating (5) over $[t,\lambda]$ we obtain:

$$x(t) - \int_t^\lambda q_1(s)ds = x(\lambda) + \int_t^\lambda x^2(s)ds.$$

Letting $\lambda \to \infty$ and utilizing our previous remarks we see that $x(t) - \int_t^\infty q_1(s)ds$ exists and is positive for each $t \in J$. Therefore the self-adjoint operator $Y(t)$ defined by

$$Y(t) \equiv X(t) + \int_t^\infty [Q(s)-Q_1(s)]ds$$

exists and is positive for each $t \in J$. Also, we have

$$\begin{aligned} R_3[Y](t) &\equiv Y' + Y^2 + Q(t) \\ &= X' + Y^2 + Q_1(t) \\ &\leq X' + X^2 + Q_1(t) \\ &= 0 \quad \text{on} \quad J \end{aligned}$$

where the inequality is due to the fact that $Y(t)$ and $X(t)$ commute and $0 < Y(t) \leq X(t)$ for each $t \in J$.

Now let $Z(t)$ be any self-adjoint solution of (3) such that $Z(a) \geq Y(a)$. By Theorem 2.1.7, $Z(t)$ exists and satisfies $Z(t) \geq Y(t) > 0$ throughout $J$. Finally, if $U(t)$ is any self-adjoint solution of (1) such that $U(a) \geq Z(a)$ then, by Theorem 4.1, $U(t)$ exists and satisfies $U(t) \geq Z(t) > 0$ throughout $J$. □

An application of Theorem 4.9 is the following:

COROLLARY 4.10 *Let* $J = [a,\infty)$, $a > 0$, *and suppose that for each* $t \in J$ *the integral* $\int_t^\infty Q(s)ds$ *exists and satisfies*

$$0 \leq \int_t^\infty Q(s)ds \leq \left(\frac{1}{4t}\right)I.$$

*Then* (1) *has a positive solution on* $J$.

Proof. The proof follows immediately from Theorem 4.9 if we take $Q_1(t) = \left(\frac{1}{4t^2}\right)I$ and $X(t) = \left(\frac{1}{2t}\right)I$. □

In our thesis we have limited ourselves to the study of existence and comparison theorems for Riccati differential equations in a $C^*$-algebra. Research has also been done on second order differential equations and oscillation theorems in $C^*$-algebras. We refer, for example, to the text by Hille ([13], Chapter 9) and to the research papers of G.J. Etgen and R.T. Lewis [8], G.J. Etgen and J.F. Pawlowski [9], T.L. Hayden and H.C. Howard [11], and C.H. williams [28].

To study oscillation theorems in a $C^*$-algebra, one of course has

to first define what a "zero" of a $C^*$-valued function is. This is done in a natural manner by defining the $C^*$-valued function $X(t)$ to have a "zero" at $t = t_o$ if it is singular at that point. Now suppose that $X(t)$ is a solution of a differential equation on $J = [a,\infty)$. We say that this solution is <u>oscillatory</u> if a) there exists a sequence $\{t_n\}$ in $J$ such that $t_n \to \infty$ and $X(t_n)$ is singular, and b) there exists at least one number $c \in J$ such that $X(c)$ is non-singular. Note that if $A$ is a self-adjoint operator such that $A \geq cI$ or $A \leq -cI$ for some $c > 0$ then $A$ is non-singular. With this in mind, we can see that Corollary 4.2, Theorem 4.4, Corollary 4.6 and Corollary 4.10 of this section give sufficient conditions for the Riccati differential equation to have solutions that have no "zeros" on $J = [a,\infty)$, $a > 0$. That is, these results can be interpreted as being non-oscillation theorems. There are still a great many of the standard oscillation and comparison theorems that have yet to be generalized to the $C^*$-algebra case.

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